The Three Commandments of Uncertainty

Advanced Uncertainties 2

Clicker Channel 1

Applications of what we learned last time

- Three laws...

1: Thou Shalt Use the Standard Deviation!

- If you have a whole bunch of repeat measurements and want to know the uncertainty

2: Three Sigma Good, One Sigma Bad!

- if you are trying to see whether your data agree with something.

3: Square before adding

- when trying to combine uncertainties

Let's go through these in turn

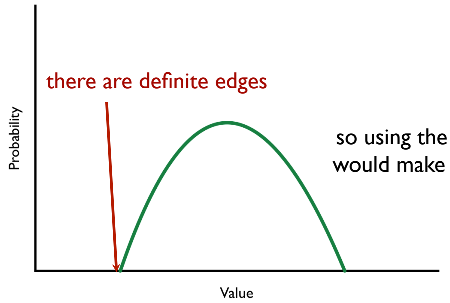
1: Thou Shalt Use the Standard Deviation!

- You often have a whole bunch of repeat measurements, and you want to know what the uncertainty is.
- e.g. I'm trying to measure how much physics a particular student knows. Each homework is a measurement of this. Scores are 7,6,7,5,8,7,6,9,7,5,6, each out of ten.
- Clearly I get a different measure of ability each time, so there must be uncertainty.

Old way to do this

- Take the highest and the lowest value and look at the difference!
- This is called the range.
- Unfortunately this is usually not a very good idea. Why?

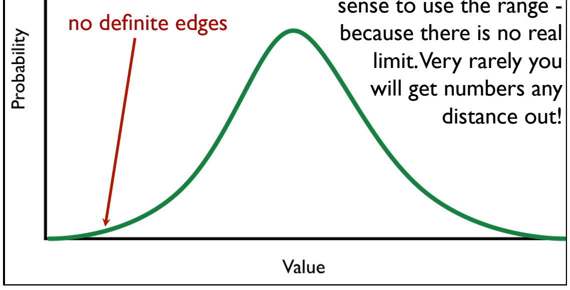
If probability distribution functions looked like this...



so using the range would make sense.

But they usually look more like this...

So it doesn't make sense to use the range - because there is no real limit. Very rarely you will get numbers any distance out!



Standard deviation

- So use the Standard Deviation instead.
- If your measurements (N of them) are x_1, x_2, x_3, ... x_n, then the standard deviation σ is given by

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

where μ is the mean.

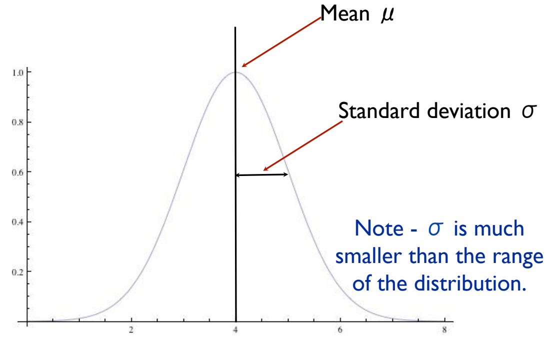
Root Mean Squared Deviation

- So let's put this in words.
- Work out how far each number is away from the mean - the "deviation"
- Square all the deviations to make them all positive.
- Average all the squared deviations
- Take square root - this is the standard deviation!

Usually done…

- in Excel or using the standard deviation function on your calculator

In this case, mean=4, standard deviation = 1



So comparing data with theory

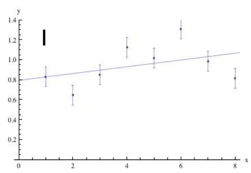
- You should expect about 68% of the measurements to lie within the standard uncertainties.
- You should expect roughly equal numbers of observations to lie above and below the theory.
- And the ones that lie above or below should not all be at one end of the data.

So you see that a lot of the time you are outside a standard deviation

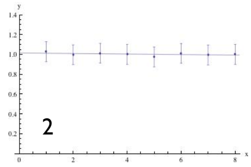
- If the curve really is a Gaussian, then (on average) 68% of measurements should fall within one standard deviation, 95% within two and 99.7% within three.
- So if I say x = 45.3±0.2, I do NOT mean that x is always in the range 45.1 to 45.5.
- Typically it will be in this range about 68% of the time.
- It will lie between 44.9 and 45.7 95% of the time.

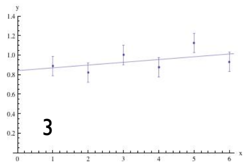
ISO standard

- There is actually international agreement that the standard deviation of the distribution of measurements is used as a measure of the uncertainty.
- It is called the "Standard uncertainty"



You are comparing your data to a model (straight line). Which fit is best?





3 is best

- 2 looks good - too good. If the uncertainties were accurate, you'd expect 32% of the time to be outside the error bars. Instead, not only does the model lie inside all the error bars, it lies close to the centre of them all.
- Clearly the error-bars are wrong - and possibly the data have been faked.

so... Thou Shalt Use the Standard Deviation!

- If you have a whole bunch of repeat measurements and want to know the uncertainty

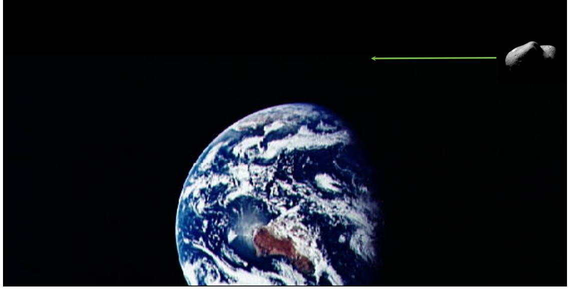
2: Three Sigma Good, One Sigma Bad!

- if you are trying to see whether your data agree with something.

Significance Test

- You've got a measurement or observation. Which has an uncertainty (standard deviation).
- Could this measurement be consistent with some other number?
- For example...

Imagine you' ve detected an asteroid. You calculate its orbit, and find that it will pass a distance 1000±500 km from the Earth. Are we safe?



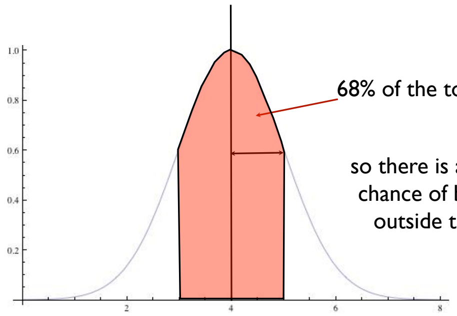
Testing Theories

- Imagine that you are a brilliant theorist. You have come up with a Grand Unified Theory, which predicts that the mass of a particular particle (in some units) should be 5.0
- An experimentalist, hoping to ruin your day, measures the mass of this particle as 4.8±0.2.
- Is your theory dead?

The rule is…

- See how many standard deviations the two numbers are apart.

One standard deviation



68% of the total area

so there is a 32% chance of being outside this.

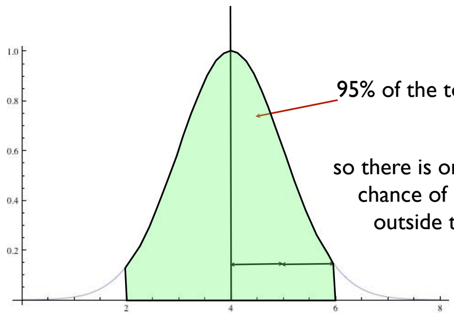
So if two numbers different by only one standard deviation…

- We call this "One Sigma", 1 σ…
- There is a 32% chance that a discrepancy this big could be purely due to chance.
- So you don't know the discrepancy is real.

e.g. Opinion Poles

- The labour party will get 47±3% of the vote.
- Will they get less than 50%? We don't know. It's likely but certainly not proven.
- Because 47% is only 1 σ away from 50%.

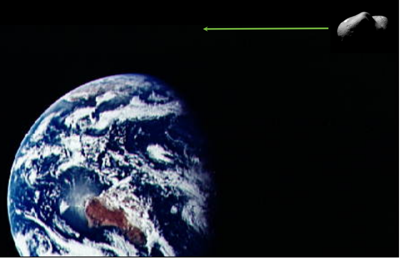
Two standard deviations



95% of the total area

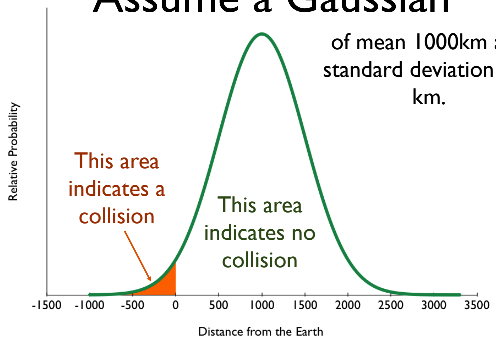
so there is only a 5% chance of being outside this.

Imagine you've detected an asteroid. You calculate its orbit, and find that it will pass a distance 1000±500 km from the Earth. Are we safe?



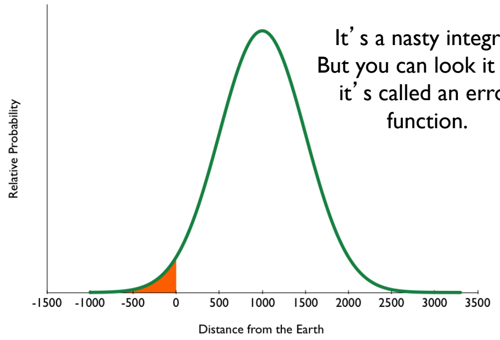
Two Sigma Difference

Assume a Gaussian



of mean 1000km and standard deviation 500 km.

Integrate...



It's a nasty integral. But you can look it up - it's called an error function.

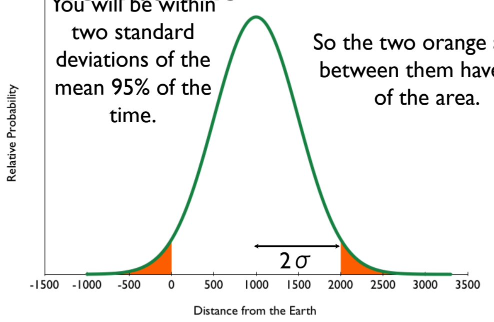
Or remember

- 68% within one standard deviation of the mean
- 95% within two standard deviations
- 99.7% within three standard deviations

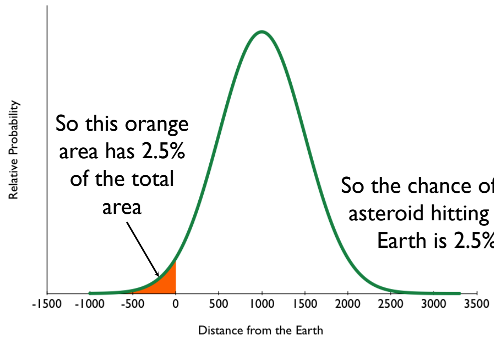
Two standard deviations

You will be within two standard deviations of the mean 95% of the time.

So the two orange areas between them have 5% of the area.



2.5%



So the chance of the asteroid hitting the Earth is 2.5%

2 σ

- Is pretty good evidence that the two numbers are different.
- It's called 95% confidence.
- Widely used in the Social Sciences

But in Physics

- we normally like things to be 3 σ different before we are prepared to believe that they really are different.
- This gives 99.7% confidence.

so...Three Sigma Good, One Sigma Bad!

- if you are trying to see whether your data agree with something.
- But beware - it only works if distributions are Gaussian.
- In the real-world, 3 σ events happen far more often than 0.3% of the time...

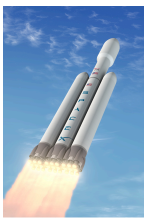
3: Square before adding

- when trying to combine uncertainties

Add up uncertainties

- Let's say you are adding together two numbers which are both uncertain.
- You might think that the uncertainty in the sum was just the two individual uncertainties added together.
- But you'd be wrong...

For example



- Example: you have a three stage rocket. Each stage increases the payload speed by 10±1 km/s.
- If the final speed is less than 26.6 km/s, the spacecraft fails to make orbit. What are the odds of this happening?

Range of speeds

- The possible range of final speeds could be large - if all three rockets did one standard deviation better than the average, your final speed would be 33 km/s.
- If all three did two standard deviations below the average, your final speed would be only 24 km/s.
- But it's not very likely that all three would perform so well or so badly together. It could happen, but odds are some would do better and some worse most of the time.

Adding in Quadrature

- If you are adding or subtracting numbers which have uncertainties...
- and if the uncertainties are independent (i.e. whether one of the numbers is above or below the mean is uncorrelated with the others)...
- Work out the final uncertainty by adding up the components IN QUADRATURE.

Square them, add them together, then take the square root.

If your final answer X is given by $x = A + B + C + \ldots$

or $x = A - B - C - \ldots$

or any other combination of plus and minus signs, then the uncertainty in x, σ_x, is given by

$$\sigma_x^2 = \sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \ldots$$

where σ_A is the uncertainty in A, σ_B is the uncertainty in B, etc. Note - the equation is the same regardless of whether you are adding or subtracting A, B C, etc

Final Speed

- So in our case, the final speed of the rocket F is equal to the speeds imparted by the three stages (let's call them A, B and C) added together.
- The uncertainty in F is thus

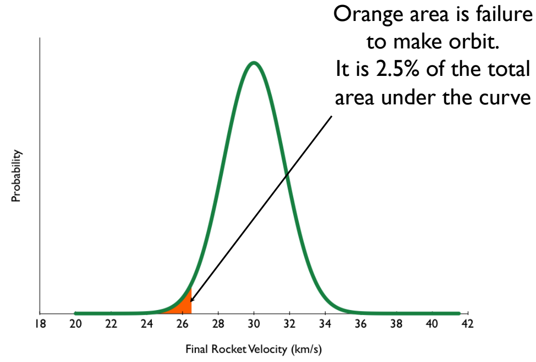
$$\sigma_F = \sqrt{\sigma_A^2 + \sigma_B^2 + \sigma_C^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} = 1.73 \text{ km } s^{-1}$$

Less than straight addition

- Adding uncertainties in quadrature gives a smaller answer than just adding them up straight.
- This is because it allows for the different uncertainties sometimes going in different directions and cancelling.

So the final speed...

- 30 ± 1.7 km/s.
- So what are the odds of it failing to make orbit (speed of less than 26.6 km/s?)
- This is two standard deviations away from the mean.
- The odds of being at least that far out are 5%.
- But we only fail to make orbit if we are low by that amount, not high.
- So chance is 2.5%



Comparing Results

- You do an experiment to prove your arch-rival wrong.
- Your rival's experiment measured a particular parameter as 10±4
- You find a value of 15±3
- Are the results inconsistent? Has one of you stuffed up?

Work out uncertainty in DIFFERENCE

- A = 10±4
- B = 15±3
- what is the uncertainty in D = A - B?

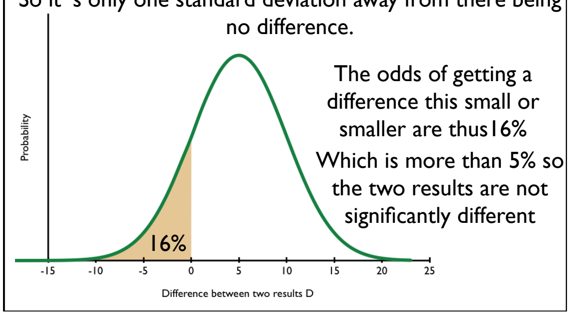
Answer

$$\sigma_D = \sqrt{\sigma_A^2 + \sigma_B^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

- So the difference is D = 5±5.

16% Chance

So it's only one standard deviation away from there being no difference.



So... Square before adding

- when trying to combine uncertainties

Consequences of Quadrature

Uncertainty propagation equations

- By using quadrature and a bit of calculus, you can come up with equations for combining uncertainties in different situations.
- These are shown in the front of the lab book

Constants

- Adding a constant to something does not change the uncertainty.
- Multiply by a constant and you need to multiply the uncertainty by the same constant.

If x = A times B, or A divided B (or B divided by A):

$$\left(\frac{\sigma_x}{x}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2$$

If $x = A^n$... $\frac{\sigma_x}{x} = n\frac{\sigma_A}{A}$

If x = Ln(A) (log to base e)... $\sigma_x = \frac{\sigma_A}{A}$

if $x = e^A$... $\frac{\sigma_x}{x} = \sigma_A$

And so on - see lab manual...

Biggest Uncertainty

- Because you square uncertainties before combining them, if one source of uncertainty is larger than the others, it totally dominates the final uncertainty.
- You can reduce the uncertainty in the minor contributors as much as you like and it will make little difference to the final uncertainty.
- So - if you want to make something more consistent, find out the biggest cause of inconsistency and work on reducing that.

For example...

- You are trying to flick peas into your neighbour's soup.
- You keep missing.
- Two sources of uncertainty - the angle at which you flick them, and the variable size of the Peas.
- Angle is far more important.
- So even spending millions of dollars on genetically modified peas with absolutely identical sizes won't help you appreciably.

Averages

- Another consequence of quadrature.
- If you take lots of measurements and average them, the average is more reliable (less uncertain) than any of the individual measurements.
- That's why you take averages!
- How does this work?

Average of some values

- Add them all up, and divide by the number of values.

$$\bar{x} = \frac{1}{N}(x_1 + x_2 + x_3 + \ldots)$$

If all the individual measurements have an uncertainty σ, then we can use quadrature to work out the uncertainty in the mean $\bar{x}$

$$\sigma_{\text{mean}} = \frac{1}{N}\sqrt{\sigma^2 + \sigma^2 + \sigma^2 + \ldots} = \frac{1}{N}\sqrt{N\sigma^2} = \frac{\sigma}{\sqrt{N}}$$

Uncertainty in the MEAN

- is the uncertainty in each of the individual measurements divided by the square root of the number of measurements.

Technicality

- (for complicated reasons, you actually divide by the square root of N-1, not the square root of N. Makes little difference when N is large enough)

$$\sigma_{\text{mean}} = \frac{\sigma_{\text{individual}}}{\sqrt{N-1}}$$

Practice...

- You'll get to practice this in the labs and in the first tutorials after the break.

Conclusions

- Standard deviation is the best way to measure an uncertainty from repeat measurements.
- If you assume that uncertainties are Gaussian, you can work out how likely various outcomes are if you know the uncertainty.
- If uncertainties are independent, you can use quadrature to combine them.

Thanks!

- for being a great class!
- Enjoy the break (and the homework...)
- Hope to see you in the Astrophysics option in PHYS1201...