Iteration

The general way to solve force/momentum problems

Course Reps Nominated

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Forces and Motion

- Before proceeding, we need to make sure we have a good understanding of forces and motion.

Clicker Question

A ball is thrown up into the air. Which of the following is a correct description of the forces acting on it?

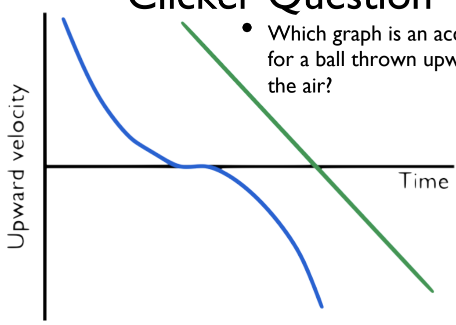
1. A steadily decreasing upward force from the moment it leaves my hand until it reaches its highest point; on the way down there is a steadily increasing downward force of gravity.
2. An almost constant downward force of gravity along with an upward force that steadily decreases until the ball reaches its highest point; on the way down there is only an almost constant downward force of gravity.
3. An almost constant downward force of gravity only.

Answer

- A constant downward force only.
- If you gave one of the other answers, you're probably confusing momentum with force.

Clicker Question

- Which graph is an accurate one for a ball thrown upwards into the air?

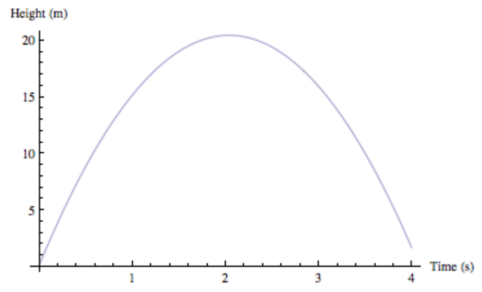


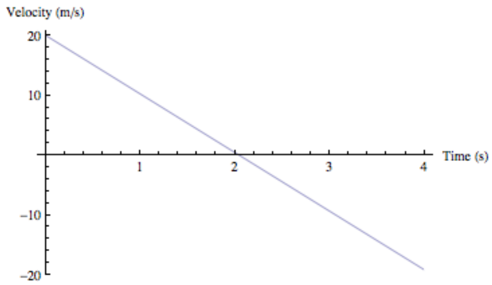
The green line

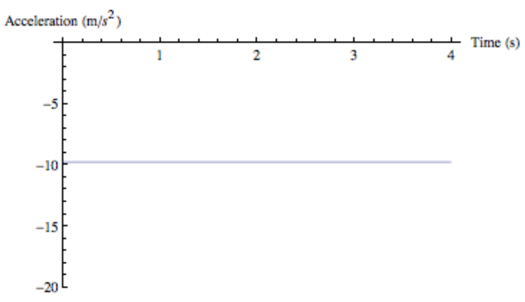
- The velocity changes at a constant rate

I throw a ball up into the air.

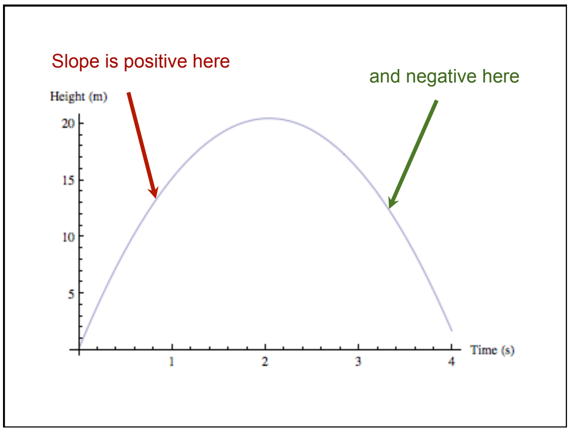
- Measure positive distance upwards.
- Sketch graphs of:
  - Position vs. time
  - Velocity vs. time
  - Acceleration vs. time

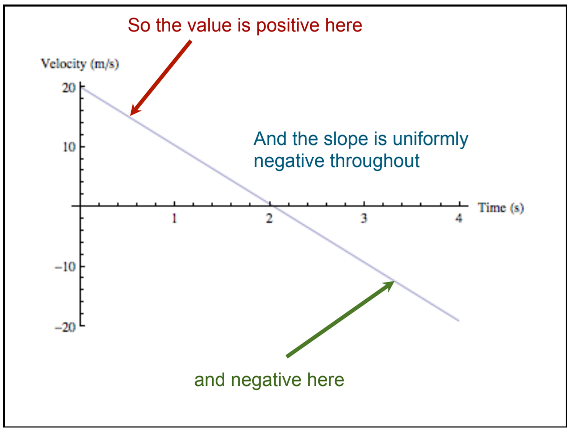


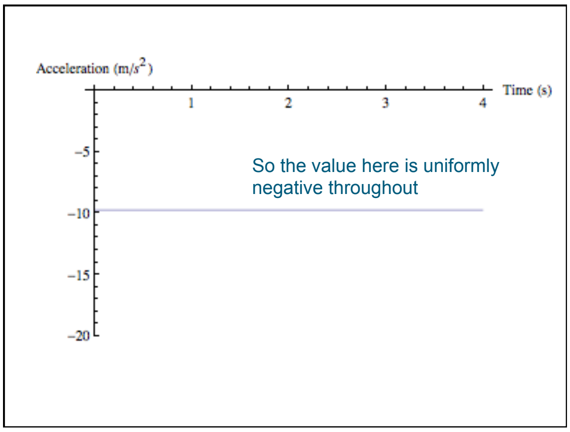




Each graph is the gradient of the previous one

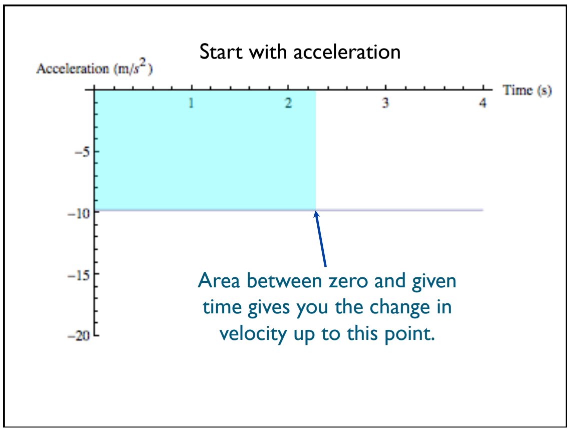


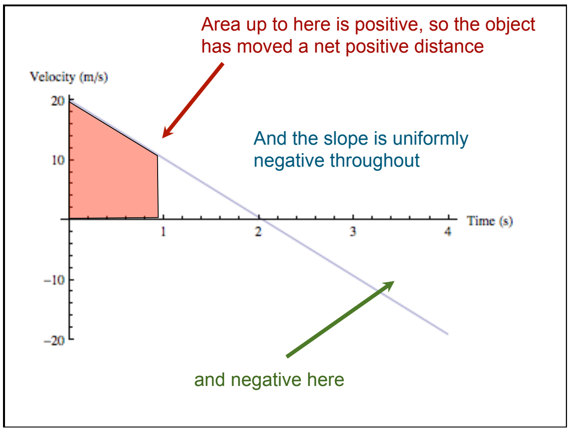


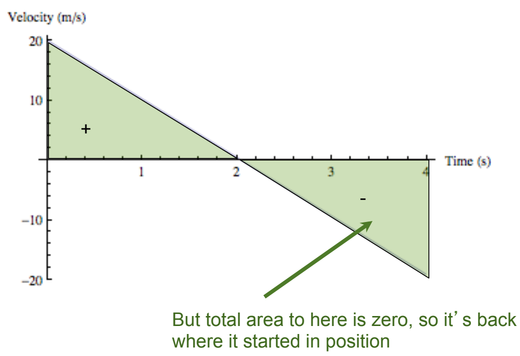


You can also go backwards

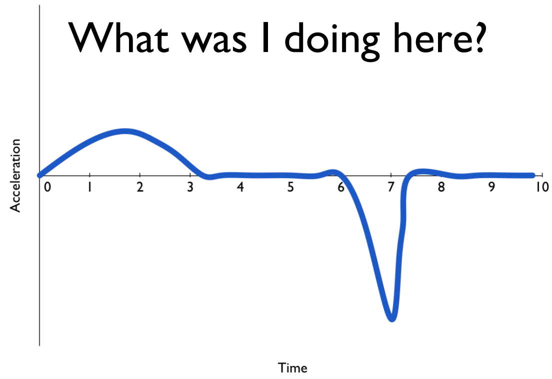
- Integrate the curve to find the change in the next one







What was I doing here?

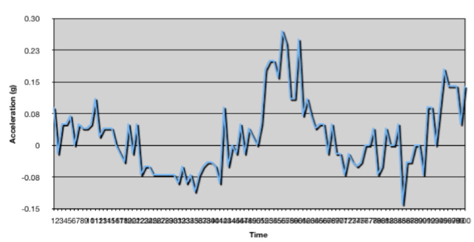


Walking into a wall…

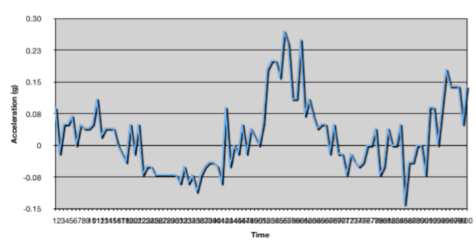
- Start at rest, accelerate for a bit, walk at a constant speed, then abruptly decelerate when I hit the wall.

What does real-world acceleration look like?

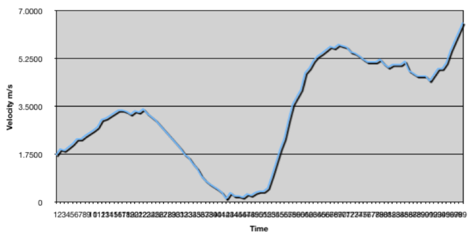
- I used the accelerometer on my iPhone to record my acceleration while driving one day (using the Accelgraph app).

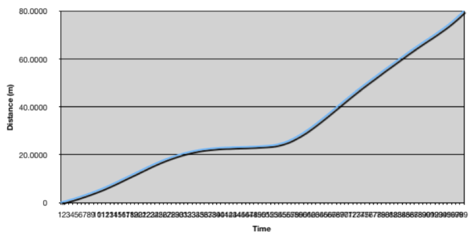


What was I doing?



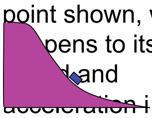
Sketch what graphs of velocity and position vs time would look like.





Question 6a

- A cart of a roller-coaster rolls down the track shown below. As the cart rolls beyond the point shown, what happens to its speed and acceleration in the direction of motion?



1. Both decrease
2. The speed decreases but the acceleration increases
3. Both remain constant
4. The speed increases, but the acceleration decreases
5. Both increase
6. Other

Still going downhill

- So in the absence of friction it will always keep speeding up.
- But - the slope is getting gentler.
- So the forward component of the gravitational force is getting smaller.
- So the acceleration is decreasing.

And now

- Given some complicated varying force, how can we work out the position?
- The answer is "Iteration"

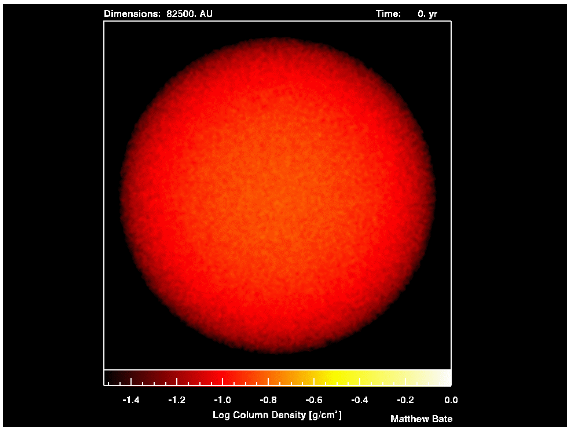
Iteration

•Solving problems step by tiny step…

Iteration

- **Iteration** means the act of repeating a process usually with the aim of approaching a desired goal or target or result. Each repetition of the process is also called an "iteration", and the results of one iteration are used as the starting point for the next iteration.

You can do some amazing things with iteration…



Movie taken from

- http://www.astro.ex.ac.uk/people/mbate/Cluster/cluster3d.html

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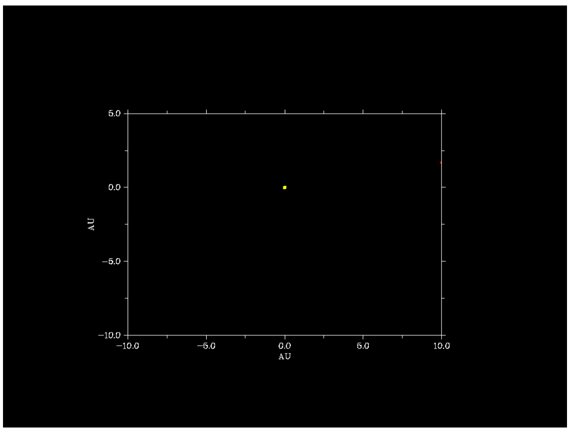
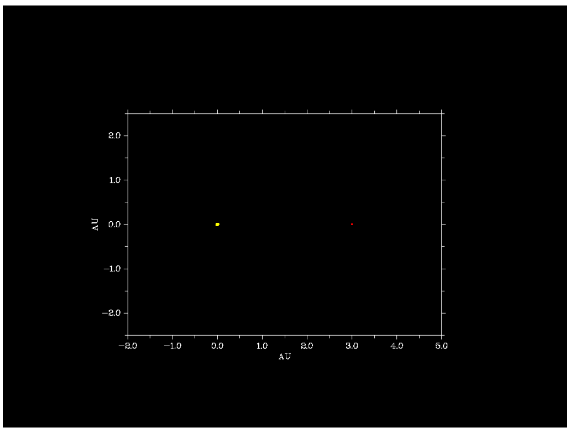
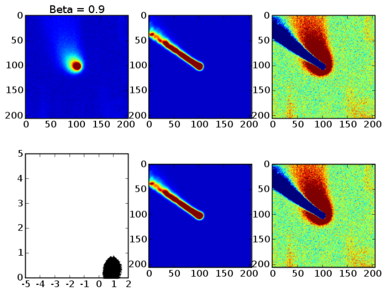






Basic idea

- Turn a big problem that we can't solve into lots and lots of small problems that we can solve. **In this case -**
- **The big problem -** calculate how something moves under the influence of ever-changing forces.
- **The small problem -** calculate how the position and speed of an object changes over a very short time.

What makes it possible

- Over a short time, you can use some sort of solvable approximation to the true motion of an object.
- If the time interval is short enough, even a crude approximation can be quite good enough.

1. Start off with a known position and velocity.
2. Use this position and velocity to work out where you will be a short time later (pretending the velocity is not changing)
3. Use the force to work out the acceleration
4. Use the acceleration to work out what the velocity will be a short time later (pretending the acceleration isn't changing).
5. Back to 1.

Distance update equation

- If your speed ***v*** is approximately constant, what is the equation for how far you go, **Δx**, in a time **Δt**?

1. $\Delta x = x\,v$
2. $\Delta x = x\,\Delta t$
3. $\Delta x = v\,\Delta t$
4. $\Delta x = v/\Delta t$
5. $\Delta x = \Delta t/v$
6. $\Delta x = v - \Delta t$
7. $\Delta x = \Delta t - v$
8. *None of the above*
9. *Not enough information given.*

Velocity update change

- If your acceleration ***a*** is approximately constant, what is the equation for how much your speed will change, **Δv**, in a time **Δt**?

1. $\Delta v = x\, v/\Delta t$
2. $\Delta v = a\, \Delta t$
3. $\Delta v = a/\Delta t$
4. $\Delta v = \Delta t/a$
5. $\Delta v = a - \Delta t$
6. $\Delta v = \Delta t - a$
7. *None of the above*
8. *Not enough information given.*

You just use the definition of velocity and acceleration

- Velocity is how far you go in a given time. So if you have time Δt, you multiply this by the velocity to find out how far you've gone Δx.
- Acceleration is how much your speed increases in a given time. So if you have time Δt, you multiply this by the acceleration to find out how much your speed has increased Δv.

This is our approximation

- We assume that the object moves at its initial speed throughout the short time interval.
- We assume that at the end of the interval its speed abruptly changes by the acceleration at the beginning of the time interval times the length of the time interval.

This method assumes that the velocity and acceleration don't change much in a timestep.

- Is this valid?
- Depends on the length of each time-step - Δt

One time step…

- At time t
- Position = x
- Velocity = v

At time $t+\Delta t$

- Position = $x+\Delta x = x + v\,\Delta t$
- Velocity = $v + \Delta v = v + a\,\Delta t$

And we also need an equation for the force - for example, the following equation is for a weight on the end of a spring.

$$F = mg - kx$$

Simplest possible…

- This approximation (speed throughout time-step equal to initial speed, speed at end of time-step equal to initial speed plus initial acceleration times time) is known as "Euler's Method".
- You can use more complicated approximations which have the benefit of being more accurate but take more computer time to evaluate - for example:

Use constant acceleration formula…

- Instead of updating x with x + v Δt
- Update it with x + v $\Delta t + 1/2\, a\, \Delta t^2$
- Still not perfectly accurate (as acceleration changes during a timestep) but probably better.
- But for the moment - let's try Euler's method (the simplest) and see where it gets us…

Let's try it.

- Before you start the iteration, you need to know:
  - The starting time, position and velocity
  - An equation for the force
- Let's do an example using a vertical spring-mass system

$$F = mg - kx$$

Let's have a mass of 0.1kg, k=5 N/m, and start at time 0, position x=0 and velocity v=0.

Keep track of some variables

- Next step - decide which variables you need to keep track of.
- In this case, it will be:
  - Time t
  - Position x
  - Velocity v
- We will choose to have x increasing downwards, so a positive force is a downwards one

Iteration Equations

- We need to write down the equations we will use in every time step, to update the variables we are tracking (t, x and v).
- These equations tell us what the value of each of these variables will be at the end of each time-step.

- For time: $t+\Delta t$
- For position: $x+v\Delta t$
- For velocity: $v+a\Delta t$
  - We get *a* from the force equation -

$$F = mg - kx = ma$$

- So…

$$a = g - \frac{k}{m}x$$

- Substituting this in, for velocity: $v + \left(g - \frac{k}{m}x\right)\Delta t$

So at each time-step, we replace

- Time t with $t+\Delta t$
- Position x with $x+v\Delta t$
- Velocity v with $v + \left(g - \frac{k}{m}x\right)\Delta t$

These are our iteration equations.

One last decision before we start

- We need to choose the time-step Δt.
- If we choose it too large, the velocity and acceleration will change too much during each time-step, making our calculation inaccurate.
- If we choose it too small, we will have to compute a vast number of time steps, which is slow.
- I'll come back to how you make the choice - for the moment, let's pick Δt = 0.1 seconds.

Let's go

- Start off with t=0, x=0, v=0
- Apply our equations:
  - New value of t is t + Δt = 0+0.1 = 0.1
  - New value of x is x + v Δt = 0+0×0.1 = 0
  - New value of v is

$$v+\left(g-\frac{k}{m}x\right)\Delta t=0+\left(9.8-\frac{5}{0.1}0\right)\times 0.1=0.98$$

So after 0.1 seconds…

- According to our method, the position hasn't changed (still zero) but the velocity has increased to 0.98 m/s.
- Now do this again, using these new numbers as the starting parameters..

Second iteration

- Start off with t=0.1, x=0, v=0.98
- Apply our equations:
  - New value of t is t + Δt = 0.1+0.1 = 0.2
  - New value of x is x + v Δt = 0+0.98×0.1 = 0.098
  - New value of v is

$$v+\left(g-\frac{k}{m}x\right)\Delta t=0.98+\left(9.8-\frac{5}{0.1}0\right)\times 0.1=1.96$$

Third iteration

- Start off with t=0.2, x=0.098, v=1.96
- Apply our equations:
  - New value of t is t + Δt = 0.2+0.1 = 0.3
  - New value of x is x + v Δt = 0.098+1.96×0.1 = 0.294
  - New value of v is

$$v+\left(g-\frac{k}{m}x\right)\Delta t=1.96+\left(9.8-\frac{5}{0.1}0.098\right)\times 0.1=2.45$$

Fourth iteration

- Start off with t=0.3, x=0.294, v=2.45
- Apply our equations:
  - New value of t is t + Δt = 0.3+0.1 = 0.4
  - New value of x is x + v Δt = 0.294+2.45×0.1 = 0.539
  - New value of v is

$$v+\left(g-\frac{k}{m}x\right)\Delta t=2.45+\left(9.8-\frac{5}{0.1}0.294\right)\times 0.1=1.96$$

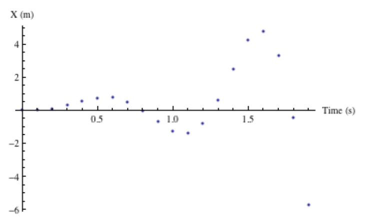
And so on…

- Do the calculations for each step, and then use the results as the input for the next step.
- That's what iteration means!
- What results do we get?

Results for first few iterations (steps)

t	x	v
0	*0*	*0*
0.1	*0*	*0.98*
0.2	*0.098*	*1.96*
0.3	*0.294*	*2.45*
0.4	*0.539*	*1.96*
0.5	*0.735*	*0.245*

A graph of the first twenty interations…



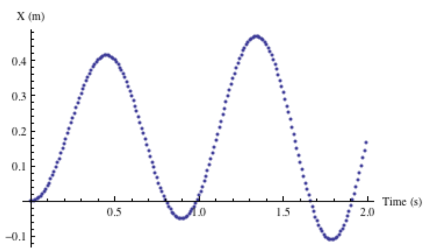
Good and bad

- If you remember - the correct solution is an oscillation.
- Our iteration has correctly produced an oscillation.
- But it has the amplitude steadily increasing - which is wrong.
- Springs don't do that!

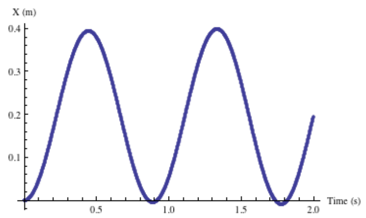
Our time step was too big.

- The approximation (that the speed and velocity are approximately constant within each timestep) wasn't good enough.
- If we make our timestep smaller… (say 0.01 sec)…
- We have to do a lot more steps…

But it gets better…



And if we make our time step smaller still - say 0.001 sec…



Really rather good…

- But I needed to do 2000 steps (iterations) to get the last plot.
- Which would have been very tedious and error-prone had I not used a computer…
- Luckily we have computers and doing those 2000 steps took less than 0.1 sec…

So even this crude approximation…

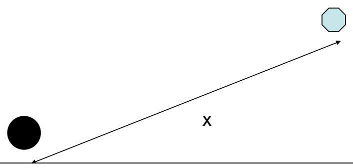
- It pretty good with small timesteps.
- And with the speed of modern computers, small timesteps are not much of a problem.
- Using a better (more complicated) approximation to the motion in each timestep will mean that you can get away with bigger timesteps.
- But each timestep needs more calculations to evaluate - so overall you may not be better off.

But it's painful

- So do it by computer!
- Example python program

Let's try an example

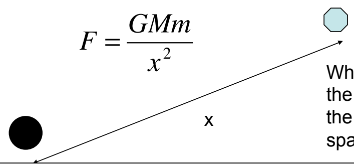
- A spaceship near a black hole…



What forces apply?

- In this case there is only one - the gravitational force.
- Being in space there is no friction or drag, so…

$$F = \frac{GMm}{x^2}$$



Where M is the mass of the black hole and m the mass of the spaceship.

What variables will we track?

- Time, position (x) and velocity (v) as before.
- For one-dimensional problems it will always be these.
- In 3D, you will need to track vector position and vector velocity (i.e the three components of each) so you will be tracking 7 numbers.

Iteration equations

- For time: t + Δt (as before)
- For position: x+ v Δt (as before)
- But what about for velocity?

$$F = \frac{GMm}{x^2}$$

Write it down…

What will the velocity be at the end of a time-step?

x increases away from the black hole. Velocity is (as always) rate of change of x.

The answer…

- Gravity works to decrease the (outwards) velocity

$$v - \frac{GM}{x^2}\Delta t$$

Let's chose some values

- Mass of the black hole = 10^{31} kg
- Starting distance = 1,000,000 km
- Starting speed = 2000 km/s away
- (You've been blasting away from it as hard as you could - but now your fuel has run out… Is your speed great enough to escape?)

Python simulation

Summary

- Divide up your problem into little tiny steps.
- Write down an approximate set of equations for each step
- Plug numbers into these formulae over and over again - taking the output from one step as the input to the next.