Propagation of Uncertainties

If you know the uncertainty in some variable A, but want to know the uncertainty in some other variable X, which is a function of A (X=F(A)), there are two methods:

Simple Method

If A = $A_0 \pm \sigma_A$, then your best guess at X will be X0=F(A0). To work out the uncertainty in X, $\sigma X = |F(A_0) - F(A_0+\sigma_A)|$ - i.e. work out the value of X when you add σ_A to A, and the difference between this and the best guess on X gives you the uncertainty in X.

Formula Method

You can use calculus to derive equations for the effect of a small change in A upon X. From this you get the following equations for propagating uncertainties in different functions.

Adding a constant

If $X = A + \mathrm{C}$, where C is a constant with negligible uncertainty, then:

$$\sigma_X = \sigma_A$$

(i.e. no change)

Multiplying by a constant

If $X = \mathrm{C} \times A$, where C is a constant with negligible uncertainty then:

$$\sigma_X = \mathrm{C}\sigma_A$$

Raising to a constant power

If $X = A^n$, and the uncertainty in n is small enough to ignore, then:

$$\frac{\sigma_X}{X} = n\frac{\sigma_A}{A}$$

Logarithms

If $X = ln(A)$ (log to the base e), then:

$$\sigma_X = \frac{\sigma_A}{A}$$

Exponential

$X = e^A$, then:

$$\frac{\sigma_X}{X} = \sigma_A$$

Combining Uncertainties

If you are combining multiple numbers, each with their own uncertainty, things are a bit different. If the different uncertainties are independent of each other, you can't just add them up, as sometimes they will cancel. In that case, use the following equations:

Sum or difference
If $X = A + B$ or $X = A - B$ then:

$$\sigma_X{}^2 = \sigma_A{}^2 + \sigma_B{}^2$$

Product or fraction
If $X = A \times B$ or $X = A/B$ then:

$$\left(\frac{\sigma_X}{X}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2$$

General case
If $X = f(A, B, C, \dots)$ then:

$$(\sigma_X)^2 = (\sigma_X^A)^2 + (\sigma_X^B)^2 + (\sigma_X^C)^2 + \cdots$$

where

$$\sigma_X^A = \left(\frac{\partial X}{\partial A}\right)\sigma_A$$

and so on. The term $(\partial X/\partial A)$ is the partial derivative of X with respect to A – think of it as the ordinary derivative of X with respect to A where you pretend that the other variables $B, C, \dots$ are constants.

Averaging
If X is the average of n different measurements $A_1, A_2, \dots A_n$, each with the same uncertainty σ_A, then:

$$\sigma_X = \frac{1}{\sqrt{n-1}}\sigma_A$$