Class 8 Notes: Ionisation and recombination II

In the previous class we understood the basic mechanics of how atoms are ionised and recombine. Our goal in this class is to put that information to use, in order to understand the equilibrium ionisation state of the ISM. We will do so in two limiting cases: when the main ionisation process is collisional, and when it is photoionisation. We will also calculate the expected observational signatures of systems in ionisation equilibrium.

I. Collisional ionisation equilibrium

First consider a hot gas where collisional ionisation is the dominant ionisation process, balanced by radiative recombination as the dominant recombination process. In equilibrium, the rates of ionisation and recombination must match for every species, so we have

$$k_{\mathrm{ci}} n_e n(X^{n+}) = k_{\mathrm{rr}} n_e n(X^{(n+1)+}), \tag{1}$$

where the k's are the rate coefficients for collisional ionisation and radiative recombination. Thus in equilibrium the ratio of the number of atoms in the two ionisation states is given by

$$\frac{n(X^{n+})}{n(X^{(n+1)+})} = \frac{k_{\mathrm{rr}}}{k_{\mathrm{ci}}}. \tag{2}$$

Recall that we showed earlier that the collisional ionisation rate coefficient when electron energies are near the ionisation threshold is approximately

$$k_{\mathrm{ci}} \approx C\pi a_0^2 \left(\frac{8k_B T}{\pi m_e}\right)^{1/2} e^{-I/k_B T}, \tag{3}$$

where I is the ionisation potential.

For radiative recombination, we can estimate the rate using the Milne relation. Again, recall that we showed that the radiative recombination rate cross section for electrons of energy E is related to the photoionisation cross section by

$$\sigma_{\mathrm{rr}}(E) = \frac{g_\ell}{g_u} \frac{(I+E)^2}{E m_e c^2} \sigma_{\mathrm{pi}}(h\nu = I + E). \tag{4}$$

As for hydrogen, we can compute the total recombination rate coefficient simply by integrating $\sigma_{\mathrm{rr}}(E)$ over a Maxwellian distribution of electron energies. However, to do this we would have to know $\sigma_{\mathrm{pi}}(\nu)$, either from theory or from laboratory measurement. For hydrogen we have the former, but for most other species we must rely on the latter.

While this is the most accurate approach, and it is the one that people take to generate detailed predictions, we can obtain reasonably good approximate results analytically. These provide us with useful rules of thumb. First, let us rewrite the photoionisation cross section in terms of an oscillator strength. Recall that the oscillator strength is related to the cross section by

$$\sigma(\nu) = \frac{\pi e^2}{m_e c} f \phi_\nu. \tag{5}$$

Thus we can define the oscillator strength $f_{\rm pi}$ for photoionisation by

$$\int_I^\infty \sigma_{\rm pi}\, d(h\nu) = \frac{\pi e^2}{m_e c} h f_{\rm pi}. \tag{6}$$

Next, recall the approximate dependence of $\sigma_{\rm pi}$ on frequency: $\sigma_{\rm pi} \propto (h\nu)^{-3}$. This leaves out features due to ionisation edges and similar effects, but it is not a bad broad-brush estimate. If we approximate $\sigma_{\rm pi}(E) = \sigma_{\rm pi}(I)(h\nu/I)^{-3}$, then we have

$$\frac{\pi e^2}{m_e c} h f_{\rm pi} = \sigma_{\rm pi}(I) I^3 \int_I^\infty (h\nu)^{-3}\, d(h\nu) = \sigma_{\rm pi}(I)\frac{I}{2} \quad \Longrightarrow \quad \sigma_{\rm pi}(I) = \frac{2\pi e^2}{m_e c} f_{\rm pi} \frac{h}{I}. \tag{7}$$

Inserting the approximation $\sigma_{\rm pi}(I+E) = \sigma_{\rm pi}(I) I^3/(I+E)^3$ into the Milne relation gives us an estimate for the radiative recombination cross section:

$$\sigma_{\rm rr}(E) = \frac{g_\ell}{g_u} \frac{(I+E)^2}{E m_e c^2} \left[\frac{2\pi e^2}{m_e c} f_{\rm pi} \frac{h}{I} \left(\frac{I}{I+E}\right)^3\right] = \frac{g_\ell}{g_u} \left(\frac{2\pi e^2 h}{m_e^2 c^3}\right) f_{\rm pi} \frac{I^2}{E(I+E)} \tag{8}$$

Using this estimate, let us compute the radiative recombination rate coefficient:

$$\begin{aligned} k_{\rm rr} &= \left(\frac{8 k_B T}{\pi m_e}\right)^{1/2} \int_0^\infty \sigma_{\rm rr}(E) x e^{-x}\, dx \qquad (9) \\ &= \frac{g_\ell}{g_u} \left(\frac{2\pi e^2 h}{m_e^2 c^3}\right) f_{\rm pi} \left(\frac{I}{k_B T}\right)^2 e^{I/k_B T} \Gamma\left(0, I/k_B T\right), \qquad (10) \end{aligned}$$

where $\Gamma(0,x)$ is an incomplete gamma function. In the limit $I \gg k_B T$, which is generally the regime where we're interested in computing collisional ionisation equilibrium (for reasons we'll see in a moment), we can Taylor expand this to

$$k_{\rm rr} \approx \frac{g_\ell}{g_u} \left(\frac{2\pi e^2 h}{m_e^2 c^3}\right) f_{\rm pi} \frac{I}{k_B T}. \tag{11}$$

Now that we have the recombination rate coefficient, we are in a position to figure out the conditions under which the gas will change from predominantly being in ionisation state X^{n+} to predominantly $X^{(n+1)+}$. Comparing the collisional ionisation and radiative rate coefficients, we have

$$\frac{n(X^{n+})}{n(X^{(n+1)+})} = \frac{k_{\rm rr}}{k_{\rm ci}} \approx 4\pi\alpha^3 \frac{g_\ell}{g_u} \frac{f_{\rm pi}}{C} \frac{I}{k_B T} e^{I/k_B T}, \tag{12}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. The temperature at which this ratio is unity, i.e. where there are equal numbers of atoms in the two ionisation states, is given implicitly by the solution to

$$\frac{I}{k_B T} e^{I/k_B T} = \frac{C}{4\pi f_{\rm pi}} \frac{g_u}{g_\ell} \frac{1}{\alpha^3} \tag{13}$$

For $C \approx 1$, $g_u \approx g_\ell$, and $f_{\rm pi} \approx 1/2$ (a typical value), the solution is $I/k_BT = 10.6$. Thus a good rule of thumb is that the gas becomes 50% ionised when the temperature reaches $k_BT = I/10$. This justifies our earlier approximation that $I \gg k_BT$. It also makes a point that is not so obvious: even at temperatures where the typical particle has a kinetic energy a factor of 10 below the ionisation potential, we still expect very significant levels of ionisation. This is because there is a tail of particles at higher energies that can create ions. Applying this result to hydrogen, for which $I/k_B = 1.6 \times 10^5$ K, we expect 50% ionisation at the substantially lower temperature of $\sim 1.6 \times 10^4$ K.

Finally, a note of caution: our calculation omitted dielectronic recombination. As a result, for those multi-electron atoms for which dielectronic recombination is significant, the ionisation fraction at a given temperature will be less – possibly significantly less – than the estimate we just derived.

II. Photoionisation equilibrium

We now consider cases where the main ionisation process is photons rather than collisions. Such regions are characteristically found around hot stars or similar sources of ionising photons, and are known as H II regions, since their dominant component is H II.

A. The Strömgrem sphere

The basic model of H II regions we will adopt was developed by Bengt Strömgren in the 1930s. We consider a uniform medium with number density of H nuclei $n_{\rm H}$, at the centre of which is placed a source of ionising radiation with a luminosity Q_0 photons s^{-1} with energies above 13.6 eV. The photons ionise the hydrogen around the source.

We can solve this problem approximately using the following very simple idea: the cross section of neutral hydrogen atoms to ionising photons is huge, so the mean free path of ionising photons through a predominantly neutral region is negligibly small. For this reason, let us approximate that the medium consist of a spherical volume centred on the source that is fully ionised, which is filled with ionising photons, and around it a medium that is fully neutral, where no ionising photons penetrate. The radius of this volume is called the Strömgren radius, R_s, and the volume is know as a Strömgren sphere.

Note that this approximation is only good if the ionising photons have energies that are not too large compared to 13.6 eV, because the cross section drops rapidly with energy. Thus the Strömgren sphere is a good model for ionisation driven by hot stars, where the ionising photon energies only go up to a few times 13.6 eV, but not for photoionisation driven by compact objects whose emission is mostly in X-rays; we will not treat the latter case in this class, but it can be handled using the same basic idea.

Returning to the Strömgren sphere case: in equilibrium every ionising photon must be absorbed within the ionised volume – otherwise it would strike the neutral gas on the border and ionise it, expanding the ionised region. The absorption is provided by neutral hydrogen atoms that are created inside the ionised volume via recombinations. We approximate that every atom created by recombination immediately encounters an ionising photon, absorbing it and ionising again. Thus in equilibrium over a given time period one neutral atom must be created for every ionising photon injected, i.e., the recombination rate throughout the ionised volume must match the ionising luminosity.

This gives us a simple condition for R_S: balancing recombinations against ionisations, we have

$$Q_0 = \frac{4}{3}\pi R_S^3 \alpha_B n_{\rm H}^2 \qquad \Longrightarrow \qquad R_S = \left(\frac{3Q_0}{4\pi\alpha_B n_{\rm H}^2}\right)^{1/3} = 9.8\times 10^{18} Q_{0,49}^{1/3} n_2^{-2/3}\ \mathrm{cm}, \tag{14}$$

where we have assumed full ionisation, so $n_{\rm H} = n_{\rm H^+} = n_e$, the numerical evaluation is for a temperature of $T = 10^4$ K (used to set α_B), $Q_{0,49} = Q_0/10^{49}\ \mathrm{s}^{-1}$, and $n_2 = n_{\rm H}/100\ \mathrm{cm}^{-3}$. The temperature is typical of H II regions, and the ionising luminosity is typical of O stars. We have also used the case B recombination coefficient, to be consistent with our assumption that ionising photons have short mean free paths. We'll check that in a moment.

First, let's check if our assumption of steady state ionisation is reasonable. Suppose that the source suddenly turns on. The time it will require to ionise the region is simply limited by the supply of ionising photons, and is roughly the number of ions to be created divided by the rate at which photons are supplied:

$$t_{\rm ion} = \frac{(4/3)\pi R_S^3 n_{\rm H}}{Q_0} = \frac{1}{\alpha_B n_{\rm H}} = \frac{1.2\ \mathrm{kyr}}{n_2}. \tag{15}$$

Should the source suddenly turn off, the time it will take the nebula to recombine is the same, since the recombination rate per proton is $\alpha_B n_{\rm H}$. In comparison, the timescale over which gas can be expected to move is the sound crossing timescale,

$$t_{\rm sound} = \frac{R_S}{c_s} = 240 Q_{0,49}^{1/3} n_2^{-2/3}\ \mathrm{kyr}, \tag{16}$$

where we have used $c_s = \sqrt{2k_B T/m_{\rm H}} = 13\ \mathrm{km\ s^{-1}}$ at $T = 10^4$ K. Thus ionisation equilibrium will be established much faster than the gas can be expected to move around.

Strömgren spheres are detectable in various ways, including by hydrogen recombination lines and by free-free emission. The emission measure averaged over the projected area of the sphere is

$$\langle \mathrm{EM}\rangle = \frac{1}{\pi R_S^2}\int_0^{R_S} 2\pi r \left(\int n_{\rm H}^2\, ds\right) dr = \frac{n_{\rm H}^2}{\pi R_S^2}\int dV = \frac{4}{3} n_{\rm H}^2 R_S. \tag{17}$$

B. Neutral gas in H II regions

The Strömgren Sphere is the 0th-order approximation to the structure of an H II region. Now let us consider a first-order approximation in which we actually solve for the ionisation structure. We continue to use the case B approximation, we continue to use spherical symmetry, we continue to assume ionisation equilibrium, and for simplicity we assume that the ionising spectrum consists of photons with median energy $h\nu$, and we neglect changes in the spectral distribution of the photons as they propagate through the H II region. The neutral cross section at this median energy is $\sigma_{\rm pi}$.

Let $Q(r)$ be the ionising photon luminosity passing through the shell at radius r. Photon conservation demands that any change in $Q(r)$ as the photons cross the shell from r to $r + dr$ be due to photon absorptions within the shell, which must occur at a rate equal to the recombination rate in order to maintain ionisation balance. The recombination rate in the shell is $4\pi r^2 \alpha_B n_{\rm H}^2 x^2 \, dr$, where $x = n_{\rm H^+}/n_{\rm H} = n_e/n_{\rm H}$. Thus we have

$$dQ = -4\pi r^2 \alpha_B n_{\rm H}^2 x^2 \, dr \qquad \Longrightarrow \qquad Q(r) - Q_0 = -4\pi n_{\rm H}^2 \alpha_B \int_0^r x^2 r'^2 \, dr'. \quad (18)$$

For simplicity we can make a change of variables $y = r/R_S$, in which case with a little re-arranging we obtain

$$Q(r) = Q_0 \left(1 - 3 \int_0^{r/R_S} x^2 y^2 \, dy \right). \quad (19)$$

The photoionisation and recombination rates must also match the absorption rate. The flux of photons passing through the shell at radius r is $Q(r)/4\pi r^2$, and the attenuation coefficient of the shell is $n_{\rm H^0}\sigma_{\rm pi} = n_{\rm H}(1-x)\sigma_{\rm pi}$. The number of absorptions per unit volume per unit time is simply the photon flux times the attenuation coefficient, so equating that with the number of recombinations per unit volume per unit time, we have

$$n_{\rm H}^2 \alpha_B x^2 = \frac{Q(r)}{4\pi r^2} n_{\rm H}(1-x)\sigma_{\rm pi}. \quad (20)$$

It is convenient to rewrite this by substituting $\alpha_B = 3Q_0/(4\pi R_S^3 n_{\rm H}^2)$ and $r = yR_S$. Doing so we obtain

$$\frac{x^2}{1-x} = \frac{Q(r)\sigma_{\rm pi}}{4\pi \alpha_B n_{\rm H}} = \frac{Q(r)}{Q_0} \frac{n_{\rm H}\sigma_{\rm pi} R_S}{3y^2} = \frac{Q(r)}{Q_0} \frac{\tau_S}{3y^2}, \quad (21)$$

where

$$\tau_S = n_{\rm H}\sigma_{\rm pi} R_S = 3400 Q_{0,49}^{1/3} n_2^{1/3} \quad (22)$$

is the optical depth to ionising photons of a column of $n_{\rm H} R_S$ neutral hydrogen atoms.

Because $\tau_0 \gg 1$, the quantity $1 - x \ll 1$, which makes sense: the neutral fraction $1 - x$ is very small. We can therefore obtain an approximate solution by setting $x = 1$ in the integral for $Q(r)$. This gives

$$Q(r) \approx Q_0 \left(1 - 3 \int_0^{r/R_S} y^2 \, dy \right) = Q_0 \left(1 - y^3 \right). \tag{23}$$

Plugging this in, we have

$$\frac{x^2}{1-x} \approx \frac{1 - y^3}{3y^2} \tau_S. \tag{24}$$

This gives the ionisation fraction x as a function of distance y, measured in units of R_S. For $y \to 0$, clearly $x \to 0$, so the ionisation fraction approaches unity at small radii. As $y \to 1$, $x \to 0$, meaning the that ionisation fraction approaches 0 at $r = R_S$, which also makes sense.

To get a sense of the typical value of x, we can compute its value at the radius that encloses half the mass or volume, which corresponds to $y = 2^{-1/3}$. At this radius, we have

$$\frac{x^2}{1-x} = \frac{1 - (1/2)}{3(1/2)^{2/3}} \tau_S = 0.26 \tau_S. \tag{25}$$

This quadratic equation is solvable, but we can be really lazy and make the equation approximately linear, since $\tau_S \gg 1$, which means $x \approx 1$. In this case we set $x^2/(1-x) = 1/(1-x)$ to first order in $1 - x$, and we have

$$1 - x \approx \frac{1}{0.26 \tau_S} \approx 1.1 \times 10^{-3} Q_{0,49}^{-1/3} n_2^{-1/3} \tag{26}$$

Thus the typical neutral fraction at the midpoint of the mass is around 10^{-3}.

C. Complications: dust and radiation pressure

There are two complications we have left out of this story, which we will not discuss in class, but which are dealt with in *Draine*: dust and radiation pressure. Dust is important because it can absorb some ionising photons. To get a sense of the importance of this, note that the dust cross section to photons near 13.6 eV is around 2×10^{-21} cm^{-2} per H atom for Milky Way dust. In contrast, the neutral hydrogen cross section at threshold is 6×10^{-18} cm^{-2}, roughly a factor of 3000 greater. On the other hand, as we have just seen the neutral fraction is only about 10^{-3}, so the cross section per H nucleus (not per H atom), is around 6×10^{-21} cm^{-2}. Of course this varies with position throughout the H II region.

Since the cross section due to hydrogen is still greater than that due to dust, even including the fact that most of the H is neutral, we are probably marginally ok neglecting dust absorption in our calculations. Indeed, our result suggests an approximate magnitude for the dust correction: since $\sigma_\mathrm{d} \approx (1/3)\sigma_\mathrm{H}$, a reasonable guess is that roughly 1/4 of the ionising photons will be absorbed by dust grains, while 3/4 will be absorbed by H atoms. This comes out surprisingly close the the detailed estimate by McKee & Williams (1997), who come up with 27% absorbed by dust. *Draine* and numerical calculations give even more detailed estimates.

Radiation pressure is a complication because it stops the ionised gas from being uniform. As we will see later on, H II regions tend to expand, but they do so at speeds slower than the sound speed within them. As a result, the gas in the H II region interior has time to spread out and become fairly uniform, so our uniform density assumption might seem reasonable.

However, the stellar radiation also exerts a force on the gas, and if this is comparable to the gas pressure, it will tend to "pile up" the ionised gas against the inner boundary of the H II region. *Draine* gives a sophisticated analysis of this in the text.

III. The hydrogen recombination spectrum

The final topic we will tackle today is what a region in ionisation equilibrium actually looks like. That is, all the recombinations that are balancing ionisations involve the emission of photons, which we can detect. What does the spectrum of those photons look like? We will not fully answer this question for another week or two, but for now we can focus on one important part of it: what does the part of the spectrum that is produced by the recombination of hydrogen, the most abundant element, look like? Knowing the answer to this question proves to be the basis for a large fraction of our understanding of star formation, among other phenomena.

A. Case A

Let's start with case A, where we assume that the region is optically thin. The rate at which recombinations create neutral hydrogen atoms in state $n\ell$ is given by $n_e n_{\mathrm{H}^+}\alpha(n\ell)$, where $\alpha(n\ell)$ is the recombination rate coefficient for that state. For $n\ell \neq 1s$, the atom will then undergo radiative decays to lower states, and these produce the line photons that we're interested in.

Let $A(n\ell \to n'\ell')$ be the Einstein A coefficient for transitions from state $n\ell$ to state $n'\ell'$, which can be computed quantum mechanically. The probability that an atom in state $n\ell$ decays to $n'\ell'$, rather than into some different state, is simply

$$\Gamma(n\ell \to n'\ell') = \frac{A(n\ell \to n'\ell')}{\sum_{n''\ell'',n''<n} A(n\ell \to n''\ell'')}. \tag{27}$$

This quantity is called the branching ratio, and is a standard quantity in particle physics. Thus, the rate at which photons corresponding to the transition $n\ell \to n'\ell'$ are emitted by atoms that recombine into state $n\ell$ is

$$n_e n_{\mathrm{H}^+}\alpha(n\ell)\Gamma(n\ell \to n'\ell'). \tag{28}$$

The emissivity is simply

$$j_\nu = \frac{h\nu}{4\pi} n_e n_{\mathrm{H}^+}\alpha(n\ell)\Gamma(n\ell \to n'\ell')\phi_\nu. \tag{29}$$

This, however, only accounts for some of the photons emitted through the $n\ell \to n'\ell'$ transition. That is because atoms in state $n\ell$ can be created through radiative decays from higher bound states, as well as through recombinations directly to state $n\ell$. Consider atoms created by recombination in the state $(n+1)\ell'$. This happens at a rate $n_e n_{\mathrm{H}^+}\alpha((n+1)\ell'')$. When these atoms decay radiatively, a fraction $\Gamma((n+1)\ell'' \to n\ell)$ will end up in state $n\ell$, and a fraction $\Gamma(n\ell \to n'\ell')$ of these will also produce $n\ell \to n'\ell'$ photons. Including this contribution, the emissivity becomes

$$j_\nu = \frac{h\nu}{4\pi} n_e n_{\mathrm{H}^+} \Gamma(n\ell \to n'\ell') \left[\alpha(n\ell) + \sum_{\ell''} \alpha((n+1)\ell'')\Gamma((n+1)\ell'' \to n\ell)\right] \quad (30)$$

Clearly this process is recursive: recombination will leave some atoms in state $(n+2)\ell''$, and some of these will end up in state $n\ell$ or $(n+1)\ell$, and therefore contribute photons. To avoid the expression spiraling out of control, we simply write the photon production rate as

$$j_\nu = \frac{h\nu}{4\pi} n_e n_{\mathrm{H}^+} \Gamma(n\ell \to n'\ell') \left[\alpha(n\ell) + \sum_{n''\ell'', n''>n} \alpha(n''\ell'')P(n''\ell'' \to n\ell)\right], \quad (31)$$

where P is the probability that an atom created in state $n''\ell''$ passes through state $n\ell$ on its way to ground. This is easy enough to compute given the known branching ratios.

B. Case B

In case B all the transition rates are the same, which one difference: photons may not be able to escape freely. Recall that case B corresponds to a nebula that is optically thick to ionising photons. The cross sections for resonant absorption of Lyman series photons decrease with n, and in the limit $n \to \infty$ the cross section is equal to the absorption cross section at threshold. This means that all the Lyman series transitions have cross sections larger than the cross section to ionising photons, and that for small n the cross sections are many orders of magnitude greater. Since both ionising photon and Lyman series photons are primarily absorbed by the same species (neutral hydrogen in the 1s state), this means that in case B the nebula must also be optically thick to Lyman series photons.

We can approximate the effects of this with the "on-the-spot" approximation that we already introduced to handle the total recombination rate. Since every Lyman series photon that is emitted is immediately reabsorbed, producing an excitation that exactly balances the de-excitation that led to photon emission, we can simply approximate the net effect by neglecting all emission in the Lyman series. In effect, we set $A(n\ell \to n'\ell') = 0$ when $n'\ell' = 1s$. The calculation of the luminosities of all the non-Lyman lines therefore proceeds in exactly the same manner as in case A, just with different branching ratios.

In case B the two strongest lines are Hα and Hβ. *Draine* gives numerical results for the rate coefficients $\alpha_{\mathrm{H}\alpha}(T)$ and $\alpha_{\mathrm{H}\beta}(T)$ for production of these two lines, and analytic approximations to them. The Hα line is particularly important because it is one of our best star formation rate indicators. This is because, as we have seen, in an H II region produced by a mass star, the recombination rate, the the Hα luminosity, and the ionising luminosity are all proportional to one another, with a constant a proportionality that depends on $\alpha_{\mathrm{H}\alpha}(T)$. By knowing this value from pure quantum mechanical theory, we can therefore compute the ionising luminosity in an H II region directly from its Hα luminosity. When we get to the star formation part of the course, we will exploit this knowledge to measure the star formation rates of galaxies.

1. 2-photon emission

   The $n = 2$ level requires special attention in case B. That is because the only transitions allowed out of this level are Lyman transitions, so in the approximation that $A(n\ell \to 1s) = 0$, all the recombined hydrogen atoms eventually accumulate in $n = 2$ states and would decay no further. This level consists of the $2s$ and $2p$ states, and we can let $\alpha_{\mathrm{eff},2s}$ and $\alpha_{\mathrm{eff},2p}$ be the effective rates for populating them – effective meaning that we include not only recombinations directly to these states, but also recombinations to higher states that eventually cascade down into $2s$ or $2p$.

   Since every recombining atom eventually winds up in one of the states, we have $\alpha_{\mathrm{eff},2s} + \alpha_{\mathrm{eff},2p} = \alpha_B$, i.e. if we sum the effective α's for the two states, it must add up to the total recombination rate. The values of $\alpha_{\mathrm{eff},2s}$ and $\alpha_{\mathrm{eff},2p}$ can be calculated by exactly the same method as all the line strengths, i.e. just by summing up branching ratios. It turns out that $\alpha_{\mathrm{eff},2s} \approx (1/3)\alpha_B$ and $\alpha_{\mathrm{eff},2p} \approx (2/3)\alpha_B$; more precise numerical values are given in *Draine*.

   We can then ask about the fate of an atom that winds up in one of these two levels. First consider what happens to atoms that end up in the $2s$ state. The transition $2s \to 1s$ is forbidden because $\Delta\ell = 0$, but it can happen, albeit at a very slow rate, $A_{2s\to 1s} = 8.23\ \mathrm{s}^{-1}$. The decay is a two-photon process, so it produces a continuous spectrum from $\nu = 0$ to $\nu = 3I_{\mathrm{H}}/4$ (the energy of the level), i.e. the sum of the two photon energies must be $3I_{\mathrm{H}}/4$, but each photon individually can have any energy between that and zero. The spectrum can be calculated quantum mechanically, and found in standard references. We let $P_\nu^{(2s)}(\nu)$ be the probability that the $2s$ decay results in one of the emitted photons being in the frequency range ν to $\nu + d\nu$; clearly energy conservation requires that $P_\nu^{(2s)}(\nu) = P_\nu^{(2s)}(\nu_{\mathrm{Ly}\alpha} - \nu)$, where $\nu_{\mathrm{Ly}\alpha} = 3I_{\mathrm{H}}/4h$. The peak is at $\nu = \nu_{\mathrm{Ly}\alpha}/2$. If every atom that enters the $2s$ state decays via this process, then the emissivity is

   $$j_\nu(2s \to 1s) = \frac{h\nu}{4\pi} n_e n_{\mathrm{H}^+} \alpha_{\mathrm{eff},2s} P_\nu^{(2s)}. \tag{32}$$

   Because the Einstein A coefficient for this decay is so small, atoms may have time time to leave the $2s$ state collisionally rather than radiatively. Collisions that take the atom to a higher state simply restart the decay process, and have no net effect. Collisions that take the atom to the $1s$ state are possible, but that rate is quite low. Instead, the main way of depopulating the $2s$ state collisionally is via collisional transitions to the $2p$ state. The rate coefficients for this transition are known; we denote them $q_{p,2s\to 2p}$ and $q_{e,2s\to 2p}$, with the first representing the rate due to collisions with protons, and the second indicating the rate due to collisions with electrons.

To account for collisions we can compute a branching ratio exactly as we did for radiative decays. The rate at which atoms leave the $2s$ state collisionally is $n_e q_{e,2s\to 2p} + n_p q_{p,2s\to 2p}$, and the rate at which they leave it radiatively is $A_{2s\to 1s}$. Thus the branching ratio for radiative decay is

$$\Gamma(2s \to 1s) \approx \frac{A_{2s\to 1s}}{A_{2s\to 1s} + n_e(q_{e,2s\to 2p} + q_{p,2s\to 2p})}, \tag{33}$$

where we have set $n_e \approx n_p$, as appropriate for a nearly fully ionised region. We usually write this in terms of a critical density. We define

$$n_{e,\mathrm{crit}} = \frac{A_{2s\to 1s}}{q_{e,2s\to 2p} + q_{p,2s\to 2p}} = 1880\ \mathrm{cm}^{-3}, \tag{34}$$

and with this definition the branching ratio becomes

$$\Gamma(2s \to 1s) = \frac{1}{1 + n_e/n_{e,\mathrm{crit}}}. \tag{35}$$

Physically, $n_{e,\mathrm{crit}}$ is simply the electron density for which the rates of radiative and collisional de-excitation are equal – we'll see this again later. With this definition, we can write the emissivity as

$$j_\nu(2s \to 1s) = \frac{h\nu}{4\pi} n_e n_{\mathrm{H}^+} \alpha_{\mathrm{eff},2s} \Gamma(2s \to 1s) P_\nu^{(2s)} = \left(\frac{h\nu}{4\pi}\right) \frac{n_e n_{\mathrm{H}^+} \alpha_{\mathrm{eff},2s}}{1 + n_e/n_{e,\mathrm{crit}}} P_\nu^{(2s)}. \tag{36}$$

Note that this has an important implication for observations: the strength of the two-photon emission spectrum depends on the density inside the H II region, and is weakest when $n_e \gg n_{e,\mathrm{crit}}$. As a result, we can use the strength of two-photon emission as a diagnostic of H II region density. Low density regions have strong two-photon emission, and high density regions have weak two-photon emission.

2. Lyman α emission

Finally, let us turn our attention to atoms that end up in the $2p$ state, either via collisional excitations from the $2s$ state, or by direct decays from higher n states. Collisional de-excitation out of this state occurs at a negligible rate, so the only decay path for these atoms is via the Lyman α transition. Given the cross-section of Lyman α absorption and assuming a Gaussian velocity dispersion, we can write the Lyman α optical depth as

$$\tau_{\mathrm{Ly}\alpha} = 8.0 \times 10^4 \left(\frac{15 \text{ km s}^{-1}}{b}\right) \tau_{\mathrm{LyC}}, \tag{37}$$

where τ_{LyC} is the Lyman continuum optical depth, i.e. the optical depth to photons at 13.6 eV. Since case B means that $\tau_{\mathrm{LyC}} > 1$, clearly $\tau_{\mathrm{Ly}\alpha}$ is immense, at least of order 10^5. Thus Lyman α photons travel only a tiny distance before being re-absorbed. The photons can eventually escape in two ways: first, they can be absorbed by a dust grain instead of a H atom. Second, each time they interact with a neutral H atom, the photon will be Doppler shifted by some small amount, depending on the atom's random velocity. This causes the photons to undergo a random walk in frequency. Eventually, they walk far enough that they are either far enough from line centre to escape, or they random walk into the frequency of another line that lies close to Lyα, most commonly O III, and are absorbed by one of those atoms. This problem is treated in detail in *Draine*.

C. Radio recombination lines

Thus far our treatment has assumed that upper states are populated only by recombinations and radiative decays, i.e., if you find an atom with an electron in, say, the $4p$ orbital, it must have gotten there because the atom recombined directly into that state, or because it recombined into a higher n state and then decayed. This is a good approximation for low to moderate n. However, for very high n, $\gtrsim 100$, there is another mechanism that can populate levels: three-body collisions between a proton and two electrons. Because the number of degenerate substates of a given electronic state n rises as n^2 (because the number of possible l values is n, and for each ℓ value there are $2\ell - 1$ distinct m_ℓ values), there is a lot of phase space available for collisional recombination into high n states, and non-negligible populations can build up at high n.

The high n atoms are rare enough that they not particularly important when it comes to the total energy or ionisation budget, but they are significant for observational reasons: transitions from $n+1$ to n, referred to as the H$n\alpha$ transition, can produce radio photons. If one works out the energy levels of the $n+1$ and n states, the frequency of the $n+1 \to n$ transition is

$$\nu_n = \frac{2n+1}{[n(n+1)]^2} \frac{I_{\mathrm{H}}}{h} \approx 6.48 \left(\frac{100.5}{n+0.5}\right)^3 \text{ GHz}. \tag{38}$$

Photons in this frequency range have the great advantage that (1) they can be detected from the ground, since the atmosphere is transparent at these frequencies, and (2) dust attenuation is essentially negligible in the radio, so this emission can still be seen even from H II regions whose optical light is completely obscured by dust. The H 166α line is particularly convenient to observe, because it just so happens to lie extremely close in frequency to the 21 cm line, so one can usually observe both at the same time.