ASTR 4020 / 8020: Diffuse Matter and Star Formation

Class 3 Notes: Radiation-matter interaction

Very broadly speaking, the last class was all about matter-matter interaction: atoms and molecules running into one another, and the rates at which they do so. This is one fundamental process in the diffuse matter of the ISM. However, a second and equally import process is the interaction of matter with radiation fields – collisions between atoms and photons, rather than between atoms and other atoms. Our goal today is to explore interactions of this type. We will do so by beginning with a brief review of radiation transfer and radiation fields, before spending the majority of the class on the statistical mechanics of matter-radiation coupling.

I. Describing the radiation field

A. The radiation intensity and its equivalents

A radiation field is most easily described in terms of the radiation intensity $I(\nu, \mathbf{n}, \mathbf{r}, t)$. The intensity gives the amount of radiant energy per unit area per unit frequency per unit solid angle. It is a function of time t, the position in space $\mathbf{r}$, the direction $\mathbf{n}$, and the frequency ν. Intuitively, we can understand what the intensity is telling us as follows: suppose we were to place a detector at some position $\mathbf{r}$ in space at some time t. This detector has a collecting area dA, and it is sensitive to light only within some narrow frequency range from ν to $\nu + d\nu$. In addition, the detector is directional: it only picks up radiation coming from a small range of solid angle $d\Omega$ about a particular direction $\mathbf{n}$. If we turn the detector on for a time interval dt, then the amount of energy dE that it receives will be

$$dE = I(\nu, \mathbf{n}, \mathbf{r}, t)\, dA\, d\nu\, d\Omega\, dt. \tag{1}$$

If a region is in local thermodynamic equilibrium (LTE – a term we will define more precisely later on), then one can show (but we will not demonstrate here – this is covered in the astrophysical processes course) that the intensity is equal to the Planck function

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} \text{ (in LTE).} \tag{2}$$

Note that this is independent of $\mathbf{n}$, because if the radiation field varies by direction, then it can't be in equilibrium.

The relationship between I_ν and $B_\nu(T)$ motivates us to define an alternative description of the radiation field. For any given intensity I_ν at a particular frequency ν, we can always solve equation 2 for T. We define the value of T that solves this equation as the *brightness temperature*,

$$T_B(\nu, \mathbf{n}, \mathbf{r}, t) = \frac{h\nu/k}{\ln[1 + 2h\nu^3/c^2 I(\nu, \mathbf{n}, \mathbf{r}, t)]}. \tag{3}$$

The brightness temperature is simply the temperature of a blackbody that produces the same intensity I as the given one. There are two things worth pointing out about this. First, although the motivation for the definition of the brightness temperature is in terms of the Planck function that describes the radiation field in LTE, one can compute the brightness temperature for any intensity, regardless of whether the system is in LTE. The brightness temperature is nothing but a mathematical expression derived from I. Second, and a corollary of the first point, is that since I is a function of ν, $\mathbf{n}$, and $\mathbf{r}$, the brightness temperature is too – that is, not only can the brightness temperature be different at different locations $\mathbf{r}$, as one might expect, can be different at different frequencies ν or different directions $\mathbf{n}$ at the *same* location.

The reason to use brightness temperature is that it becomes very convenient for systems that are in or close to LTE. We will show below that for an optically thick system in LTE, the brightness temperature is simply equal to the gas temperature. This is most commonly the case in the world of radio observations, where $h\nu \ll k_B T$. In this limit we can expand the exponential in equation 2 to first order and obtain a slightly simpler expression,

$$T_A(\nu, \mathbf{n}, \mathbf{r}, t) = \frac{c^2}{2k\nu^2} I(\nu, \mathbf{n}, \mathbf{r}, t). \tag{4}$$

This is called the antenna temperature, since it comes from radio, but it is really nothing more than the first-order expansion of the brightness temperature in the low-frequency limit.

Finally, the description that we will use most often in this course is the photon occupation number,

$$n_\gamma(\nu, \mathbf{n}, \mathbf{r}, t) = \frac{c^2}{2h\nu^3} I(\nu, \mathbf{n}, \mathbf{r}, t), \tag{5}$$

This is simply I multiplied by a constant, so that it is dimensionless. Physically, it measures the number of photons per polarisation mode at a given point. It is the natural quantum mechanical description of the radiation field, since in quantum mechanics the radiation field may be considered a harmonic oscillator, and n_γ just corresponds to the quantum number describing its oscillation. The reason for using n_γ is that it makes the statistical calculations we will undertake below look simple, as it gets rid of various extraneous constants.

B. Moments of the radiation intensity

The intensity or its equivalents describe all there is to know about the radiation field (except polarisation – we could have two different intensities for two different polarisations, but for simplicity we'll neglect that). Often this is more information than we need, and instead we care only about certain averages of the intensity. One such commonly-used average is the frequency-integrated intensity

$$I = \int I_\nu \, d\nu. \tag{6}$$

This is the quantity that is relevant when we don't care about the frequency of the radiation. In some other circumstances we might not care about the direction $\mathbf{n}$, so we integrate over it. We define

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega \tag{7}$$

as the directionally-averaged intensity.

These averages are closely related to more familiar physical quantities. In particular, suppose that we are interested in the energy density in the radiation field. This is given by

$$u_\nu = \frac{1}{c} \int I(\nu, \mathbf{n}, \mathbf{r}, t) \, d\Omega = \frac{4\pi}{c} J_\nu. \tag{8}$$

You can verify for yourself that this has the correct units. To understand where it comes from intuitively, first note that we clearly want to average over directions, since we only want to know the total energy density at a point. To understand where the $1/c$ factor comes from, suppose that instead of photons we were describing some other particle traveling at half the speed of light. We are holding the intensity fixed, so that the same number of ergs per second of these particles pass a given point – they're just traveling half as fast. Clearly they must therefore be bunched up twice as closely as the photons, so the density will be twice as high. This is the effect that the $1/c$ captures.

Similarly, the next two moments of the radiation field are

$$\begin{aligned} F_\nu &= \int I(\nu, \mathbf{n}, \mathbf{r}, t) \, \mathbf{n} \, d\Omega & (9) \\ P_\nu &= \frac{1}{c} \int I(\nu, \mathbf{n}, \mathbf{r}, t) \, \mathbf{n} \otimes \mathbf{n} \, d\Omega. & (10) \end{aligned}$$

These are the radiation flux and the radiation pressure tensor.

II. Einstein coefficients

A. Definition

Now that we have introduced a description of the radiation field, we are ready to introduce the way that radiation interacts with matter. A key consideration here is that, since the matter we are concerned with is diffuse gases, almost all of the interactions are going to be resonant interactions that are associated with a change in the quantum state of the atoms or molecules in the matter. We therefore need a language to talk about such transitions. This basic language was invented by Albert Einstein, and is called the Einstein coefficients.

Consider a particle of species X with lower and upper energy levels X_ℓ and X_u, with energies E_ℓ and E_u. If a member of this species is in state ℓ, it can absorb a photon and transition to state u:

$$X_\ell + h\nu \rightarrow X_u, \qquad h\nu = E_u - E_\ell. \tag{11}$$

Suppose we now have a population of members of species X in state X_ℓ with number density n_ℓ, and that this population interacts with a population of photons of intensity I_ν. Photons with frequencies near $h\nu = E_u - E_\ell$ can be absorbed. We define the line profile function ϕ_ν as giving the relative probability that a photon of frequency ν will be absorbed, and we normalise so that

$$\int \phi_\nu \, d\nu = 1. \tag{12}$$

For now we can think of ϕ_ν as closely approximating a δ function at frequency $\nu_{u\ell} = (E_u - E_\ell)/h$, and we will calculate its true form later.

As with any other collisional process, the rate at which collisions between particles and photons produce transitions must be proportional to the number density of target particles times the number density of photons in the beam. Thus we write the rate at which photons are absorbed is

$$\left(\frac{dn_u}{dt}\right)_{\text{abs.}} = -\left(\frac{dn_\ell}{dt}\right)_{\text{abs.}} = n_\ell B_{\ell u} \overline{J}, \text{ where } \overline{J} = \int J_\nu \phi_\nu \, d\nu, \tag{13}$$

since the photon number density at frequencies near ν where they can be absorbed is proportional to $\overline{J}$. $B_{\ell u}$ is the rate coefficient for this absorption. In cgs it has units of cm^2 erg^{-1}. This type of rate coefficient has a special name: $B_{\ell u}$ is the Einstein absorption coefficient for this transition.[1]

$B_{\ell u}$ can be calculated quantum mechanically using a semi-classical approach; one treats the radiation field classically, as an oscillating electrical and magnetic field, and uses perturbation theory to compute the probability of the atom undergoing a change in state due to the perturbation. In practice these calculations are analytically tractable only for the very simplest atoms, and are numerically tractable only for slightly more complex ones. For most complex multi-electron atoms and molecules, however, the absorption rate must be measured in the lab.

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[1] **Important caution:** there are two different conventions for how to define the Einstein B coefficient. The one we have adopted here is to define it in terms of the angle-averaged intensity J_ν, which is the convention used in *Krumholz*, and also in textbooks such as *Rybicki & Lightman* and *Shu*. *Draine*, however, uses the opposite convention, which is to define B in terms of the radiation energy density u_ν. As a result, *Draine*'s B coefficients differ from the ones in these notes by a factor of $c/4\pi$. This will not matter much in practice, because we will write all results below in terms of the A coefficient that we will define momentarily, and that does not contain this ambiguity. Just be aware that two conventions exist, since many papers in the literature unfortunately do not say explicitly which one they are using.

In addition to absorption, two other types of transition are possible. First, a particle in the excited state u can spontaneously decay to state ℓ, emitting a photon:

$$X_u \rightarrow X_\ell + h\nu. \tag{14}$$

The rate at which this happens per unit volume simply depends on the number density of particles that can emit:

$$-\left(\frac{dn_u}{dt}\right)_{\text{spon. emiss.}} = \left(\frac{dn_\ell}{dt}\right)_{\text{spon. emiss.}} = n_u A_{u\ell}, \tag{15}$$

where $A_{u\ell}$, which we introduced in the last class, is a constant with units of s^{-1}. It is called the Einstein spontaneous emission coefficient, or just the Einstein A coefficient.

Finally, there is another emission process. Just as a time-varying electromagnetic field can induce a particle in state ℓ to transition to state u by perturbing it, the reverse is true: a particle in state u can be induced to transition to state ℓ:

$$X_u + h\nu \rightarrow X_\ell + 2h\nu. \tag{16}$$

The extra photon has the same direction (and phase and polarization) as the one that induced the emission. This process is known as stimulated emission. The stimulated emission rate must have the same functional form as the absorption rate, since it is essentially the same process in reverse. Thus we write

$$-\left(\frac{dn_u}{dt}\right)_{\text{stim. emiss.}} = \left(\frac{dn_\ell}{dt}\right)_{\text{stim. emiss.}} = n_u B_{u\ell} \overline{J}, \tag{17}$$

Here $B_{u\ell}$ has the same units as $B_{\ell u}$, and is called the Einstein stimulated emission coefficient.

B. Relationships

The three Einstein coefficients are not independent of one another. We can see this using the same trick as we did for collisions of material particles: considering a system in LTE. In LTE, the distribution of particle energy levels must be given by the Boltzmann distribution, while the radiation intensity has to follow the Planck function (which is just the generalisation of the Boltzmann distribution for bosons). For the matter this implies

$$n_u = \frac{g_u}{g_\ell} e^{-h\nu/kT} n_\ell, \tag{18}$$

and for the radiation

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}, \tag{19}$$

where $B_\nu(T)$ is the Planck function. Clearly this is independent of direction, so $J_\nu = I_\nu$. If we approximate ϕ_ν as a δ function, so that we ignore the variation in I_ν over it, then we have

$$\overline{J} = \int J_\nu \delta(\nu)\, d\nu = \frac{2h\nu_{u\ell}^3}{c^2} \frac{1}{e^{h\nu_{u\ell}/kT} - 1}. \tag{20}$$

In LTE, we require that the rate of change of n_u and n_l be zero, and substituting in for I_ν and n_u using LTE, we have

$$\begin{aligned} 0 &= \left(\frac{dn_u}{dt}\right)_{\text{abs.}} + \left(\frac{dn_u}{dt}\right)_{\text{stim. emiss.}} + \left(\frac{dn_u}{dt}\right)_{\text{spon. emiss.}} & (21) \\ &= n_\ell B_{\ell u} \frac{2h\nu_{u\ell}^3}{c^2} \left[\frac{1}{e^{h\nu_{u\ell}/kT} - 1}\right] - \\ & \qquad n_\ell \frac{g_u}{g_\ell} e^{-h\nu_{u\ell}/kT} \left(A_{u\ell} + B_{u\ell} \frac{2h\nu_{u\ell}^3}{c^2} \left[\frac{1}{e^{h\nu_{u\ell}/kT} - 1}\right]\right). & (22) \end{aligned}$$

This equation is required to hold independent of T. For $h\nu_{u\ell}/kT \ll 1$, the exponentials approach 1, so the terms in square brackets are very large, and we can ignore the $A_{u\ell}$ term in comparison. This immediately shows us that

$$B_{\ell u} = \frac{g_u}{g_\ell} B_{u\ell}. \tag{23}$$

Similarly, for $h\nu/kT \gg 1$, the terms in square brackets become very small, so we can drop the $B_{u\ell}$ term in comparison to the $A_{u\ell}$ one. Doing so and solving, we find

$$B_{u\ell} = \frac{c^2}{2h\nu_{u\ell}^3} A_{u\ell}. \tag{24}$$

Thus the value of $A_{u\ell}$ and the degeneracies and energies of the two levels fully determines all the Einstein coefficients.

For convenience we sometimes define the dimensionless directionally-averaged photon occupation number:

$$\langle n_\gamma \rangle = \frac{c^2}{2h\nu_{u\ell}^3} J_\nu, \tag{25}$$

where the brackets indicate that we are dealing with a quantity that has been averaged over directions. This quantity has the virtue that it allows us to express the emission and absorption rates very simply:

$$\left(\frac{dn_\ell}{dt}\right)_{\text{spon. emiss.}} = n_u A_{u\ell} \quad \left(\frac{dn_\ell}{dt}\right)_{\text{stim. emiss.}} = n_u \langle n_\gamma \rangle A_{u\ell} \quad \left(\frac{dn_u}{dt}\right)_{\text{abs.}} = \frac{g_u}{g_\ell} n_\ell \langle n_\gamma \rangle A_{u\ell}. \tag{26}$$

This definition makes clear that stimulated emission is unimportant when the photon occupation number is $\ll 1$, and dominant when it is $\gg 1$.

III. Cross sections and line profiles

It is often convenient to recast the absorption process in terms of a cross section, using something like the collision formalism we developed earlier in the class. Since each photon has an energy $h\nu$, the number of photons per unit time with frequencies from ν to $\nu+d\nu$ passing a given point is $J_\nu/h\nu$. Thus, following our example of writing reaction rates as number density times number density times cross section times velocity, we define the cross section $\sigma_{\ell u}(\nu)$ by

$$\left(\frac{dn_u}{dt}\right)_{\text{abs.}} = n_\ell \int \sigma_{\ell u}(\nu) \frac{4\pi J_\nu}{h\nu}\, d\nu \approx n_\ell J_\nu(\nu_{u\ell}) \frac{4\pi}{h\nu_{u\ell}} \int \sigma_{\ell u}(\nu)\, d\nu. \tag{27}$$

Here we are integrating over all photons energies or frequencies, and the factor of 4π is inserted because we are integrating over all directions as well. In the second step, we have assumed that $\sigma_{\ell u}(\nu)$ is very narrowly peaked around the frequency $\nu = (E_u - E_\ell)/h$, so that $J_\nu/h\nu$ is nearly constant over the range where $\sigma_{\ell u}(\nu)$ has any appreciable value, and we can take it out of the integral. This is almost always the case, unless the radiation field varies extremely rapidly with frequency.

If we now equate our formula for $(dn_u/dt)_{\text{abs}}$ in terms of $\sigma_{\ell u}$ with our formula with that in terms of the Einstein coefficients, and do a little re-arranging, we obtain

$$\int \sigma_{\ell u}(\nu)\, d\nu = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}, \tag{28}$$

where $\nu_{u\ell} = (E_u - E_\ell)/h$ is the frequency corresponding to the exact energy difference between the levels. We therefore define the line profile function ϕ_ν by

$$\sigma_{\ell u}(\nu) = \frac{g_u}{g_\ell} \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}\phi_\nu, \qquad \int \phi_\nu\, d\nu = 1. \tag{29}$$

The function ϕ_ν contains all the information about how $\sigma_{\ell u}$ depends on frequency. Thus we see that the line profile function we introduced earlier is just a representation of how the microphysical absorption cross section depends on frequency.

Similarly, considering stimulated emission gives

$$n_u J_\nu(\nu_{u\ell}) \frac{4\pi}{h\nu} \int \sigma_{u\ell}(\nu)\, d\nu = \left(\frac{dn_\ell}{dt}\right)_{\text{stim. emiss.}}, \tag{30}$$

so we can write

$$\sigma_{u\ell}(\nu) = \frac{c^2}{8\pi\nu_{u\ell}^2} A_{u\ell}\phi_\nu = \frac{g_\ell}{g_u}\sigma_{\ell u}(\nu) \tag{31}$$

Note that we are implicitly assuming that the line profile function ϕ_ν is the same for absorption and stimulated emission. To see that this must be true, simply note that, in LTE, the rates of stimulated plus spontaneous emission must balance the rate of spontaneous emission at every frequency, and that this must be true independent of the temperature, which of course changes the functional form of ϕ_ν. This is only possible if all three rates have the same frequency dependence.

A. Natural broadening

One might think that $\sigma_{\ell u}(\nu)$ should be an infinitely sharp δ function; after all, how can a photon whose energy does not precisely match the energy difference between the two levels be absorbed? However, that ignores the uncertainty principle: one cannot precisely determine the photon energy or the exact velocity of the emitting particle. It also ignores the fact that, even in the absence of quantum uncertainty, for a population of particles at finite temperature there will be a range of velocities, and thus the Doppler effect will allow emission and absorption of a range of frequencies.

First let's consider the intrinsic quantum effect, which is called natural broadening. The exact line profile this produces can be computed quantum mechanically (a full treatment is given in *Rybicki & Lightman*), but to good approximation we can write it in a form that resembles the strength of response of a system to driving near a resonance, which varies as the square of the difference between the driving and resonant frequencies. This is the Lorentz profile:

$$\phi_\nu \approx \frac{4\gamma_{u\ell}}{16\pi^2(\nu-\nu_{u\ell})^2+\gamma_{u\ell}^2}. \tag{32}$$

Here $\gamma_{u\ell}$ has units of frequency. The full width at half max of this profile is

$$(\Delta\nu)_{\mathrm{FWHM}} = \frac{\gamma_{u\ell}}{2\pi}. \tag{33}$$

What is the quantity $\gamma_{u\ell}$? We can estimate this from the uncertainty principle. The lifetimes of the upper and lower states are

$$\tau_u = \left(\sum_{j<u} A_{uj}\right)^{-1} \qquad \tau_\ell = \left(\sum_{j<\ell} A_{\ell j}\right)^{-1}. \tag{34}$$

The uncertainty principle states that $\Delta E\,\Delta t \geq \hbar$, or equivalently in terms of photon frequency $\Delta\nu\,\Delta t \geq 1/2\pi$. Since $\Delta t \sim \tau_u$ for the upper level, it follows that $\Delta\nu \sim 1/\tau_u = \sum_{j<u} A_{uj}$, and similarly for the lower one. In fact, a precise calculation gives

$$\gamma_{u\ell} = \sum_{j<u} A_{uj} + \sum_{j<\ell} A_{\ell j}. \tag{35}$$

Thus we can compute the natural linewidth from the Einstein A's.

It is sometimes convenient to think of this width in terms of velocity: what Doppler shift would be required to produce the same shift in frequency? This is just (for non-relativistic motion)

$$(\Delta v)_{\mathrm{FWHM}} = c\frac{(\Delta\nu)_{\mathrm{FWHM}}}{\nu_{u\ell}} = \frac{\lambda_{u\ell}\gamma_{u\ell}}{2\pi}, \tag{36}$$

where $\lambda_{u\ell} = c/\nu_{u\ell}$. Typical linewidths for allowed UV and optical transitions are ~ 0.01 km s^{-1}, while for X-ray transitions they can reach ~ 10 km s^{-1}. The most prominent example is Lyman α, which has $(\Delta v)_{\mathrm{FWHM}} = 0.0121$ km s^{-1}.

B. Doppler broadening and the Voigt profile

The second main source of line broadening in the context of the ISM is Doppler broadening. The effect is simply that the gas has a non-zero velocity dispersion, so there are a range of atom velocities, each producing a different Doppler shifted frequency of emission or absorption. In fact, except for X-ray lines, it is almost always the case that the Doppler width is much greater than the natural width we have just computed – since gas is usually moving around much faster than ~ 0.01 km s^{-1}.

For gas with a Maxwellian velocity distribution, the fraction f_v of particles with velocity between v and $v + dv$ is

$$f_v = \frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-(v-v_0)^2/2\sigma_v^2}, \tag{37}$$

where v_0 is the mean velocity and $\sigma_v = \sqrt{kT/m}$ is the velocity dispersion. For convenience we sometimes use the broadening parameter $b = \sqrt{2}\sigma_v$ in place of σ_v. Since the Doppler shift is simply $\nu_{u\ell} v/c$, the corresponding line profile is

$$\phi_\nu = \frac{1}{\sqrt{2\pi\sigma_\nu^2}} e^{-(\nu-\nu_0)^2/2\sigma_\nu^2}, \tag{38}$$

where $\sigma_\nu = (\sigma_v/c)\nu_{u\ell}$ and $\nu_0 = \nu_{u\ell}(1 - v_0/c)$.

In reality, both Doppler broadening and natural broadening operate at the same time. Every particle at a given velocity emits a line that is naturally broadened. As a result, the true line profile is a convolution of the Doppler and Lorentz profiles:

$$\phi_\nu = \frac{1}{\sqrt{2\pi\sigma_v^2}} \int_{-\infty}^{\infty} e^{-v^2/2\sigma_v^2} \frac{4\gamma_{u\ell}}{16\pi^2(\nu - (1 - v/c)\nu_{u\ell})^2 + \gamma_{u\ell}^2}\, dv, \tag{39}$$

This is known as the Voigt profile. Note the factor of $(1-v/c)$ in the denominator of the Lorentz profile function, representing the Doppler shift of a particular particle. For simplicity we have dropped the v_0, since we can always choose to shift our rest frame to one in which the gas is at rest.

The integral cannot be evaluated analytically in general, but we can approximate it for the most common case $\gamma_{u\ell} \ll (\sigma_v/c)\nu_{u\ell}$, i.e. where the Doppler width is much greater than the natural width. In this case the Lorentz profile is much more sharply peaked than the Doppler profile near $v = 0$, so for small velocities we can approximate it by a a δ function, i.e.

$$\frac{4\gamma_{u\ell}}{16\pi^2[\nu - (1 - v/c)\nu_{u\ell}]^2 + \gamma_{u\ell}^2} \approx \delta(v - c[1 - \nu/\nu_{u\ell}]) \tag{40}$$

In this case the integral is trivial, and reduces to simply the Doppler profile we derived earlier.

However, note that the Doppler profile falls off as $(\nu - \nu_{u\ell})^{-2}$ for frequencies far away from $\nu_{u\ell}$, whereas the Maxwellian profile falls off as e^{-v^2}, which is much faster. For frequencies far from $\nu_{u\ell}$, this means that we can instead think of the $e^{-v^2/2\sigma_v^2}$ term as a $\delta(v)$. In this case the integral is again trivial, and reduces to the Lorentz profile.

Thus we see that the shape of the Voigt profile is simply a "core" that looks like a Doppler profile, but with broad "wings" that fall off as $(\nu - \nu_{u\ell})^{-2}$, rather than $e^{-(\nu-\nu_{u\ell})^2/2\sigma_\nu^2}$, as a pure Doppler profile would. We can roughly estimate the velocity for which the transition between these two shapes occurs by solving for the velocity / frequency at which the two line profiles are equal. This is given by

$$\frac{4\gamma_{u\ell}}{16\pi^2 (v/c)^2 \nu_{u\ell}^2 + \gamma_{u\ell}^2} = \frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-v^2/2\sigma_v^2}, \tag{41}$$

where in the Lorentz profile we have written the difference in frequency $\nu - \nu_{u\ell}$ as $(v/c)\nu_{u\ell}$. For convenience we define the Doppler broadening parameter

$$b = \sqrt{2}\sigma_v, \tag{42}$$

and if we let $z = v/b$, then the velocity of the core / wings transition is given implicitly by

$$e^{z^2} = \frac{4\pi^{3/2} b}{\gamma_{u\ell}\lambda_{u\ell}} z^2 + \frac{\gamma_{u\ell}}{4\pi b}. \tag{43}$$

Since $b \gg \gamma_{u\ell}$ for almost all astrophysical applications, we can generally drop the last term. This still leaves a transcendental equation that cannot be solved analytically, but the solution is reasonably well approximated by

$$z^2 \approx 10.31 + \ln\left[\frac{7618 \text{ cm s}^{-1}}{\gamma_{u\ell}\lambda_{u\ell}} b_6\right], \tag{44}$$

where we have normalized to the value of $\gamma_{u\ell}\lambda_{u\ell}$ for Lyman α, and $b_6 = b/10$ km s^{-1}. Since the logarithmic term generally isn't large (unless we're dealing with gas at X-ray temperatures), we see that tend the damping wings dominate the profile for $|z| \gtrsim 3.2$, i.e. for velocities of more than about 4.5 times the velocity dispersion.

IV. Radiative transfer

We are now in a position to discuss the propagation of a beam of radiation through a material medium, and the interactions that take place as the photons move through the matter.

A. The transfer equation

Consider a beam of radiation of intensity I_ν (where we've dropped the argument list for conciseness) entering a slab of material of thickness ds. On the far side of the slab, the intensity that emerges is $I_\nu + dI_\nu$. The equation of radiative transfer states that

$$dI_\nu = -I_\nu \kappa_\nu \, ds + j_\nu \, ds, \tag{45}$$

where j_ν is the emissivity of the material (with units of power per unit volume per unit frequency per unit solid angle), and κ_ν is the attenuation coefficient (with units of 1/length).

Suppose now that the dominant emission and absorption processes are line emission and absorption by atoms and molecules. We can write the emissivity and attenuation coefficient in terms of the theory we have developed for these processes. For simplicity, let us continue to consider a single species with upper and lower states u and ℓ, and number densities n_u and n_ℓ for particles in that state.

For emission, recall that the rate of spontaneous decays per unit volume from state u is $n_u A_{u\ell}$ integrated over all frequencies. The rate at a specific frequency ν is $n_u A_{u\ell}\phi_\nu$. Each emission produces a photon of energy $h\nu$. Thus the power radiated per unit volume is $n_u A_{u\ell} h\nu$. In the frame comoving with the emitting particles, the emission is isotropic, and thus is evenly directed over 4π sr. Thus the emissivity is

$$j_\nu = \frac{1}{4\pi} n_u A_{u\ell} h\nu \phi_\nu. \tag{46}$$

If the material is moving this should be corrected for both Doppler shifting and beaming, although the latter is usually unimportant for non-relativistic flows.

For attenuation, we must compute the net rate of absorption, i.e. absorption minus stimulated emission. We can also do this in terms of the Einstein coefficients. The rate at which stimulated emission produces new photons of frequency ν traveling in direction $\mathbf{n}$ is $n_u B_{u\ell}(I_\nu/4\pi)\phi_\nu$ photons per unit time per unit frequency. Each photon carries energy $h\nu$. Similarly, the rate at which absorption removes photons from the beam is $n_\ell B_{\ell u}(I_\nu/4\pi)\phi_\nu$. Thus the net absorption minus emission rate is

$$\kappa_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u}\phi_\nu - \frac{h\nu}{4\pi} n_u B_{\ell u}\phi_\nu. \tag{47}$$

Note that this has the right units: 1/length. The I_ν is not included because of the way κ_ν is defined. Recalling the relationship between the B coefficients from last class, we can rewrite this as

$$\kappa_\nu = n_\ell \frac{h\nu}{4\pi} B_{\ell u} \left(1 - \frac{g_\ell}{g_u}\frac{n_u}{n_\ell}\right)\phi_\nu. \tag{48}$$

The combination that appears inside the parentheses has a specific name. In LTE, Boltzmann's law tells us that

$$n_u = \frac{g_u}{g_\ell} n_\ell e^{-E_{u\ell}/kT} \qquad \Longrightarrow \qquad \frac{g_\ell}{g_u}\frac{n_u}{n_\ell} = e^{-E_{u\ell}/kT}. \tag{49}$$

We therefore define the excitation temperature of two levels by

$$e^{-E_{u\ell}/kT_{\mathrm{exc}}} = \frac{g_\ell}{g_u}\frac{n_u}{n_\ell}. \tag{50}$$

Clearly in LTE we have $T_{\text{exc}} = T$, but out of LTE this need not hold. With this definition, the attenuation coefficient becomes

$$\kappa_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u} \left(1 - e^{-E_{u\ell}/kT_{\text{exc}}}\right) \phi_\nu. \tag{51}$$

B. Integrating the transfer equation: formal solution and uniform media

It is often convenient to make a change of variables in the transfer equation by letting

$$d\tau_\nu = \kappa_\nu \, ds, \tag{52}$$

which turns the transfer equation into

$$dI_\nu - I_\nu d\tau_\nu + S_\nu \, d\tau_\nu, \tag{53}$$

where

$$S_\nu = \frac{j_\nu}{\kappa_\nu}. \tag{54}$$

The equation can be solved formally by isolating the I_ν:

$$dI_\nu + I_\nu \, d\tau_\nu = S_\nu \, d\tau_\nu. \tag{55}$$

If we then multiply by e^{τ_ν} on both sides, we can integrate the equation:

$$\begin{aligned}
e^{\tau_\nu} \left(dI_\nu + I_\nu \, d\tau_\nu\right) &= e^{\tau_\nu} S_\nu \, d\tau_\nu && (56)\\
d\left(e^{\tau_\nu} I_\nu\right) &= e^{\tau_\nu} S_\nu \, d\tau_\nu && (57)\\
e^{\tau_\nu} I_\nu(\tau_\nu) - I_\nu(0) &= \int_0^{\tau_\nu} e^{\tau'} S_\nu \, d\tau' && (58)\\
I_\nu(\tau_\nu) &= I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau)'} S_\nu \, d\tau' && (59)
\end{aligned}$$

It's worth pausing to understand the physical meaning of this equation. Formally, it tells us the intensity along some particular ray. On this ray we have marked some starting point and labelled it optical depth 0, and we want to compute the intensity I_ν at some optical depth τ_ν further along the ray. This has two parts. The first is the intensity at the starting point, decreased by a factor of $e^{-\tau_\nu}$. It is the radiation entering the slab and being attenuated by it. The second term is an integral over the radiation that is added by emission within the slab, but also attenuated by it – radiation from the back of the slab is attenuated by more than the radiation from the front.

The physical meaning of this equation is perhaps easiest to understand by considering some special cases. Consider an infinite slab of matter in LTE at temperature T. The energy levels must therefore have an excitation temperature $T_{\text{exc}} = T$, and the radiation field must be equal to the Planck function $I_\nu = B_\nu(T)$. Since the intensity does not change anywhere, it follows that

$$dI_\nu = 0 = B_\nu \, d\tau_\nu + S_\nu \, d\tau_\nu \qquad \Longrightarrow \qquad S_\nu = \frac{j_\nu}{\kappa_\nu} = B_\nu(T). \tag{60}$$

This is another example of the trick we've been using for several classes: we have deduced this equality from LTE considerations, which ultimately come from nothing more than counting arguments. However, note that $S_\nu = j_\nu/\kappa_\nu$ is a function only of local properties of the matter, and thus must hold locally at every point. This is known as Kirchoff's Law: for matter in LTE, the emissivity and attenuation coefficients are related by

$$\frac{j_\nu}{\kappa_\nu} = B_\nu(T). \tag{61}$$

If we substitute this into the formal solution to the transfer equation, we have

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu-\tau)'} B_\nu(T_{\text{exc}})\, d\tau'. \tag{62}$$

Note that T_{exc} can be a function of position, since Kirchoff's law applies locally. However, if the temperature inside the slab is constant, the $B_\nu(T_{\text{exc}})$ comes out of the integral, and we can evaluate it trivially:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + B_\nu(T_{\text{exc}})(1-e^{-\tau_\nu}). \tag{63}$$

The physical meaning of this becomes clear if we consider how the equation behaves in the limit of large and small τ_ν. For small τ_ν, the $e^{-\tau_\nu}$ factor is unity, so the first term is $I_\nu(0)$, and the second is zero. Thus the radiation is the same as what it was on the far side of the slab. On the other hand, for large τ_ν the first term goes to zero and the second one dominates. This says that the radiation field simply approaches the Planck function for temperature $T = T_{\text{exc}}$. Thus as the radiation passes through the matter, it becomes thermalized.

C. Masers

An interesting phenomenon is possible when matter is out of LTE in a specific way. Under some conditions collisions or radiative processes can lead to a population inversion, meaning that $n_u/g_u > n_\ell/g_\ell$. In other words, there are more particles in the upper state than one would expect for a Boltzmann distribution at any temperature. Formally, in fact, in this case the excitation temperature $T_{\text{exc}} < 0$.

Consider what happens as radiation moves through matter in which a population inversion exists. We cannot use the form of the transfer equation that applies in LTE, since of course the gas cannot be in LTE if a population inversion exists. Instead, recall that we showed earlier that

$$\kappa_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u}\left(1 - \frac{g_\ell}{g_u}\frac{n_u}{n_\ell}\right) = \frac{h\nu}{4\pi} n_\ell B_{\ell u}\left(1 - e^{-E_{u\ell}/kT_{\text{exc}}}\right). \tag{64}$$

If a population inversion exists, then the term in parenthesis is negative, and the attenuation coefficient is positive. For simplicity, let us consider matter of negligible emissivity and constant, negative excitation temperature. In this case the transfer equation becomes

$$dI_\nu = -I_\nu \, d\tau_\nu \qquad \Longrightarrow \qquad I_\nu = I_\nu(0) e^{-\tau_\nu} \tag{65}$$

but with the twist that $d\tau_\nu = -\kappa_\nu \, ds$ is negative as one proceeds along the ray, so the total optical depth is negative as well. This means that the intensity increases rather than decreases exponentially as radiation propagates through the matter.

This is known as a maser or laser (microwave or light amplified by stimulated emission of radiation) because the physical origin of the effect is that stimulated emission adds new photons to the beam faster than absorptions remove them. In some astrophysical situation the $e^{-\tau_\nu}$ factor can be very large, and as a result the intensity can be huge. For some astrophysical sources the brightness temperature exceeds 10^{11} K.