ASTR 4020 / 8020: Diffuse Matter and Star Formation

Class 2 Notes: Fluid dynamics

In the last class we developed a basic theory of particle-particle collisions. In this class we will use that theory to answer the question of under what circumstances we can think of the ISM as a fluid that can be characterised by a temperature and an equation of state. We will then make some general observations about what type of fluid the ISM is, which will inform our treatment of it in the rest of the class.

I. Is the ISM a fluid?

To start with the question of whether the ISM is a fluid, we have to first recall what it means for something to be a fluid. If we were to use a vacuum pump to suck all the our of this room, leaving behind only two molecules, clearly it would be meaningless to talk about the room as a fluid. For example, there is no single density or velocity that describes the two molecules. The reason that it makes sense to talk about the air in the room as a fluid is that there are no many molecules, colliding so often, that particle velocities get randomised. In this case it makes sense to talk about the air moving at a certain velocity, because the distribution of particle velocities will be some mean plus some isotropic distribution about that mean. So the key process is randomisation: to have a fluid, particles must collide often enough for their velocity distributions to become random and isotropic.

A. Neutral particles

To evaluate this condition, we will start with the simplest case: a set of identical neutral particles, say hydrogen atoms. Since any scattering between two such particles is enough to redirect their motion in an arbitrary direction, we can consider things a fluid on scales significantly larger than the particle mean free path. For neutral particles undergoing hard-sphere scattering, this length is

$$\lambda_{\mathrm{mfp}} = \frac{1}{n\sigma} = 55 \left(\frac{r}{1\ \text{Å}}\right)^{-2} \left(\frac{n}{1\ \mathrm{cm}^{-3}}\right)^{-1}\ \mathrm{AU}, \tag{1}$$

where r is the particle radius. We have normalised here to the mean density of the ISM in the Galaxy, so this provides a useful answer: we can think of the ISM as a fluid as long as we are considering length scales larger than ~ 100 AU. This is a very tiny scale by interstellar standards, so a fluid description is appropriate.

For a population containing particles of very disparate masses the question is more complicated, because a massive particle will not change its momentum much per encounter with much less massive particles. Instead, it will take many encounters to change the massive particle's direction significantly. We will consider this case in the practice problem.

B. Charged particles

Another case that is more complicated is the case of collisions between ionised particles, because in that case, as we have seen, the cross section is not a well-defined quantity due to the long range Coulomb interaction. Instead, we must ask about how long it takes the particle's momentum to change significantly as a result of all those Coulomb interactions.

Consider a particle of charge $Z_1 e$ moving through a field of particles of charge $Z_2 e$. We have seen that the change in particle 1's transverse momentum due to an encounter at impact parameter b and velocity v is

$$\Delta p_\perp = 2\frac{Z_1 Z_2 e^2}{b v_1}. \tag{2}$$

On average the momentum change due to the many particles in the field will sum to zero, but the RMS change will not be zero – the particle's transverse momentum will undergo a random walk. We can compute the rate at which it increases by multiplying the rate at which particle 1 encounters field particles times $(\Delta p_\perp)^2$.

The rate of encounters with cross section b is $n_2 v_1 \times 2\pi b\, db$, i.e., number of targets in the field times velocity with which particle 1 moves through them times area with impact parameter between b and $b + db$. Here we're making the simplifying assumption that v_1 is much larger than the mean velocity of the field particles, so that we don't have to worry about integrating over the Maxwellian distribution they present, i.e., $v_1 \gg \overline{v}$; properly integrating over relative velocities just introduces a factor of order unity difference. Thus we get a rate of change for $(\Delta p_\perp)^2$

$$\left\langle \frac{d}{dt} \left(\Delta p_\perp\right)^2 \right\rangle = \int_{b_{\rm min}}^{b_{\rm max}} n_2 v_1 2\pi b \left(\frac{2 Z_1 Z_2 e^2}{b v_1}\right)^2 db = \frac{8\pi n_2 Z_1^2 Z_2^2 e^4}{v_1} \int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b}. \tag{3}$$

Clearly we cannot take $b_{\rm min} = 0$ or $b_{\rm max} = \infty$ without the integral diverging. For the minimum impact parameter to consider, we can adopt the impact parameter for which the impulse approximation used to compute $\Delta p_\perp$ fails. If the initial kinetic energy in the center of mass frame is E, this failure occurs when the interaction energy is comparable to E – if this is the case, then clearly we cannot ignore the deflection of the particles during the encounter. Thus we take $b_{\rm min} = Z_1 Z_2 e^2/E$. For the maximum, the plasma will shield charges on size scales longer than the Debye length,

$$L_D = \left(\frac{kT}{4\pi n_e e^2}\right)^{1/2}, \tag{4}$$

where n_e is the free electron density, and this will cut off the Coulomb force from larger distances. Thus we take $b_{\rm max} = L_D$, and we have

$$\left\langle \frac{d}{dt} \left(\Delta p_\perp\right)^2 \right\rangle = \frac{8\pi n_2 Z_1^2 Z_2^2 e^4}{v_1} \ln \Lambda, \tag{5}$$

where $\ln \Lambda$ is known as the Coulomb logarithm, and has the value

$$\ln \Lambda = \ln \left[\frac{E}{kT} \frac{(kT)^{3/2}}{(4\pi n_e)^{1/2} Z_1 Z_2 e^2} \right] = 22.1 + \ln \left[\left(\frac{E}{kT} \right) \left(\frac{T}{10^4 \text{ K}} \right)^{3/2} \left(\frac{\text{cm}^{-3}}{n_e} \right) \right] \quad (6)$$

Note that we made some very rough approximations in computing $b_{\min}$ and $b_{\max}$, but these enter the result only logarithmically.

We are now in a position to answer the question of on what length scales we can treat a plasma as a fluid. The characteristic time for the random walk produced by lots of little kicks $\Delta p_\perp$ to randomise an initial velocity v_1, known as the deflection time, is

$$t_{\text{defl}} = \frac{(m_1 v_1)^2}{\langle (d/dt)(\Delta p_\perp)^2 \rangle} = \frac{m_1^2 v_1^3}{8\pi n_2 Z_1^2 Z_2^2 e^4 \ln \Lambda}, \quad (7)$$

and the distance the particle travels in this time, the effective mean free path, is

$$\lambda_{\text{mfp}} = v_1 t_{\text{defl}} = \frac{m_1^2 v_1^4}{8\pi n_2 Z_1^2 Z_2^2 e^4 \ln \Lambda}. \quad (8)$$

If we consider electrons being deflected either by other electrons or by protons, and plug in a velocity corresponding to a kinetic energy $(3/2)kT$ (not fully consistent with our choice to take v_1 much larger than the thermal energy, but we're only after an order of magnitude estimate here), we have

$$\lambda_{\text{mfp}} = 5 \times 10^{12} \left(\frac{m_1}{m_e} \right)^2 \left(\frac{T}{10^4 \text{ K}} \right)^2 \left(\frac{0.1 \text{ cm}^{-3}}{n_2} \right) \left(\frac{25}{\ln \Lambda} \right) \text{ cm}. \quad (9)$$

Thus on size scales larger than ~ 1 AU, for densities of ~ 0.1 cm^{-3} and temperatures $\sim 10^4$ K, we may consider the electrons in a plasma to be a fluid.

II. Does the ISM have a temperature?

We have now established conditions for thinking of the ISM as a fluid. However, this does not necessarily mean that it has a well-defined temperature. Temperature is a meaningful concept only for systems in or close to thermal equilibrium, meaning that the distribution of energies follows the Maxwell-Boltzmann distribution. Does the ISM satisfy this condition?

To answer this, instead of asking about the deflection of a particle by its random walk in momentum space, we can now ask about how its energy changes due to these deflections. For equal-mass neutral particles that interact as hard spheres, every encounter leads to a significant change in energy, so the time to change a particle's energy is essentially the same as the time to change it's momentum, and the conditions for being a fluid and having a well-defined temperature are very similar.

The situation is more complicated for charged particles of unequal mass, which is an important case, since this is what happens when we have electrons and ions. Suppose particle 1 has initial velocity v_1, so its kinetic energy is $(1/2)m_1 v_1^2$, and again assume

v_1 is much greater than the thermal velocities of the field particles. Each time particle 1 undergoes an encounter, it gives momentum $\Delta p_\perp$ to the field particle, and as a result its energy decreases by $(\Delta p_\perp)^2/(2m_2)$. The time required for such encounters to completely deplete particle 1's excess energy and make it into a field particle is

$$t_{\mathrm{loss}} = \frac{m_1 v_1^2}{\langle (d/dt)(\Delta p_\perp)^2/m_2 \rangle} = \frac{m_1 m_2 v_1^3}{8\pi n_2 Z_1^2 Z_2^2 e^4 \ln \Lambda}. \tag{10}$$

If we ask how long it will take encounters between fast electrons with a speed corresponding to a temperature T_e, and a sea of protons with number density n_p, to slow down the electrons, we get

$$t_{\mathrm{loss}} = 0.4 \times \left(\frac{T_e}{10^4\ \mathrm{K}}\right)^{3/2} \left(\frac{\mathrm{cm}^{-3}}{n_p}\right) \left(\frac{25}{\ln \Lambda}\right)\ \mathrm{Myr}. \tag{11}$$

Thus hot electrons make take ~ 1 Myr to slow down due to encounters with protons.

This timescale is short enough compared to flow timescales in the ISM that for most places in the ISM it is reasonable to assume a single temperature. However, there are exceptions, for example in regions where the electrons are very hot or the density is low. In these regions, the electrons will thermalise with respect to each other quickly, but may not reach the same temperature as the protons for a long time. A gas in this state is referred to as a two-temperature plasma, because the electrons and protons each have a well-defined Maxwellian velocity distribution, but at different temperatures.

III. Fluid mechanics of the ISM

Now that we have established that we can for the most part think of the ISM as a fluid with a well-defined temperature, we can use the equations of fluid mechanics to investigate its behaviour. We will not derive these equations here – that is covered in the astrophysical processes class. We will simply assert them and work with them.

A. The conservation equations

We will start with the case of a non-magnetised fluid, and then build up to include magnetic fields. Both magnetised and unmagnetised fluid are governed by a series of conservation laws. The most basic one is conservation of mass:

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot (\rho \mathbf{v}). \tag{12}$$

This equation asserts that the change in mass density at a fixed point is equal to minus the divergence of density times velocity at that point. Physically, this is very intuitive: density at a point changes at a rate that is simply equal to the rate at which mass flows into or out of an infinitesimal volume around that point.

We can write a similar equation for conservation of momentum:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla P + \rho \nu \nabla^2 \mathbf{v}. \tag{13}$$

Note that the term $\mathbf{v} \otimes \mathbf{v}$ here is a tensor product. This is perhaps more clear if we write things out in index notation:

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_j}(\rho v_i v_j) - \frac{\partial}{\partial x_i} P + \rho\nu \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j} v_i\right) \tag{14}$$

The intuitive meaning of this equation can be understood by examining the terms one by one. The term $\rho\mathbf{v}$ is the density of momentum at a point. The term $\nabla \cdot (\rho\mathbf{v} \otimes \mathbf{v})$ is, in analogy to the equivalent term in the conservation of mass equation, the rate at which momentum is advected into or out of that point by the flow. The term ∇P is the rate at which pressure forces acting on the fluid change its momentum. Finally, the last term, $\rho\nu\nabla^2\mathbf{v}$, is the rate at which viscosity redistributes momentum; the quantity ν is called the kinematic viscosity.

The last term, the viscosity one, requires a bit more discussion. All the other terms in the momentum equation are completely analogous to Newton's second law for single particles. The viscous term, on the other hand, is unique to fluids, and does not have an analog for single particles. It describes the change in fluid momentum due to the diffusion of momentum from adjacent fluid elements. We can understand this intuitively: a fluid is composed of particles moving with random velocities in addition to their overall coherent velocity. If we pick a particular fluid element to follow, we will notice that these random velocities cause some of the particles that make it up to diffuse across its boundary to the neighbouring element, and some particles from the neighbouring element to diffuse into the one we are following. The particles that wander across the boundaries of our fluid element carry momentum with them, and this changes the momentum of the element we are following. The result is that momentum diffuses across the fluid, and this momentum diffusion is called viscosity.

B. The Reynolds Number and the Mach Number

To understand the relative importance of terms in the momentum equation, it is helpful to make order of magnitude estimates of their sizes. Let us consider a system of characteristic size L and characteristic velocity V. The natural time scale for flows in the system is L/V, so we expect time derivative terms to be of order the thing being differentiated divided by L/V. Similarly, the natural length scale for spatial derivatives is L, so we expect spatial derivative terms to be order the quantity being differentiated divided by L. If we apply these scalings to the momentum equation, we expect the various terms to scale as follows:

$$\frac{\rho V^2}{L} \sim \frac{\rho V^2}{L} + \frac{\rho c_s^2}{L} + \rho\nu\frac{V}{L^2}, \tag{15}$$

where c_s is the isothermal sound speed, and we have written the pressure as $P = \rho c_s^2$. Canceling the common factors, we get

$$1 \sim 1 + \frac{c_s^2}{V^2} + \frac{\nu}{VL}. \tag{16}$$

From this exercise, we can derive two dimensionless numbers that are going to control the behaviour of the equation. We define the Mach number and the Reynolds number as

$$\mathcal{M} \sim \frac{V}{c_s} \tag{17}$$

$$\mathrm{Re} \sim \frac{LV}{\nu}. \tag{18}$$

The meanings of these dimensionless numbers are fairly clear from the equations. If $\mathcal{M} \ll 1$, then $c_s^2/V^2 \gg 1$, and this means that the pressure term is important in determining how the fluid evolves. In contrast, if $\mathcal{M} \gg 1$, then the pressure term is unimportant for the behaviour of the fluid. Similarly, the Reynolds number is a measure of how important viscous forces are. Viscous forces are significant for $\mathrm{Re} \sim 1$ or less, and are unimportant of $\mathrm{Re} \gg 1$. We can think of the Reynolds number as describing a characteristic length scale $L \sim \nu/V$ in the flow. This is the length scale on which diffusion causes the flow to dissipate energy. Larger scale motions are effectively dissipationless, while smaller scales ones are damped out by viscosity.

So what are $\mathcal{M}$ and Re in the ISM? Well, it depends on which phase of the ISM we are considering. Let us begin with $\mathcal{M}$, which requires knowledge of the characteristic flow speed and the characteristic sound speed. The isothermal sound speed of a gas is

$$c_s = \sqrt{\frac{k_B T}{\mu m_\mathrm{H}}} = 9.1\mu^{-1/2} \left(\frac{T}{10^4\ \mathrm{K}}\right)^{1/2}\ \mathrm{km\ s^{-1}}. \tag{19}$$

where μ is the mean particle mass in units of the hydrogen mass m_H. Recall that the temperature varies from as high as $\sim 10^7$ K in the hot ionised medium, to as low as ~ 10 K in cold molecular gas. The mean particle mass will also be lower in the hot gas: a fully ionised gas has $\mu = 0.61$ due to the presence of free electrons, while a gas where all the hydrogen is in form of $\mathrm{H_2}$ has $\mu = 2.3$. Thus the sound speed will vary from ~ 500 km s^{-1} in the hottest gas to ~ 0.2 km s^{-1} in the coldest. Typical velocities vary too: hot regions are typically observed to have flow speeds of hundreds to thousands of km s^{-1}, while in atomic and molecular regions speeds tend to be closer to a few to 10 km s^{-1}. Comparing these numbers to the sound speed, we see that warmer phases of the ISM tend to have $\mathcal{M} \sim 1$ (transsonic), while colder phases have $\mathcal{M} \gg 1$ (highly supersonic).

To estimate Re, we must know the viscosity. For an ideal gas, the kinematic viscosity is $\nu = 2\overline{u}\lambda_\mathrm{mfp}$, where $\overline{u}$ is the RMS particle speed (which is of order c_s) and λ_mfp is the particle mean free-path. Using equation 1 for the mean free path and equation 19 for the sound speed (and thus roughly $\overline{u}$), we have

$$\nu \sim 10^{21} \left(\frac{T}{10^4\ \mathrm{K}}\right)^{1/2} \left(\frac{n}{1\ \mathrm{cm^{-3}}}\right)^{-1}\ \mathrm{cm^2\ s^{-1}}. \tag{20}$$

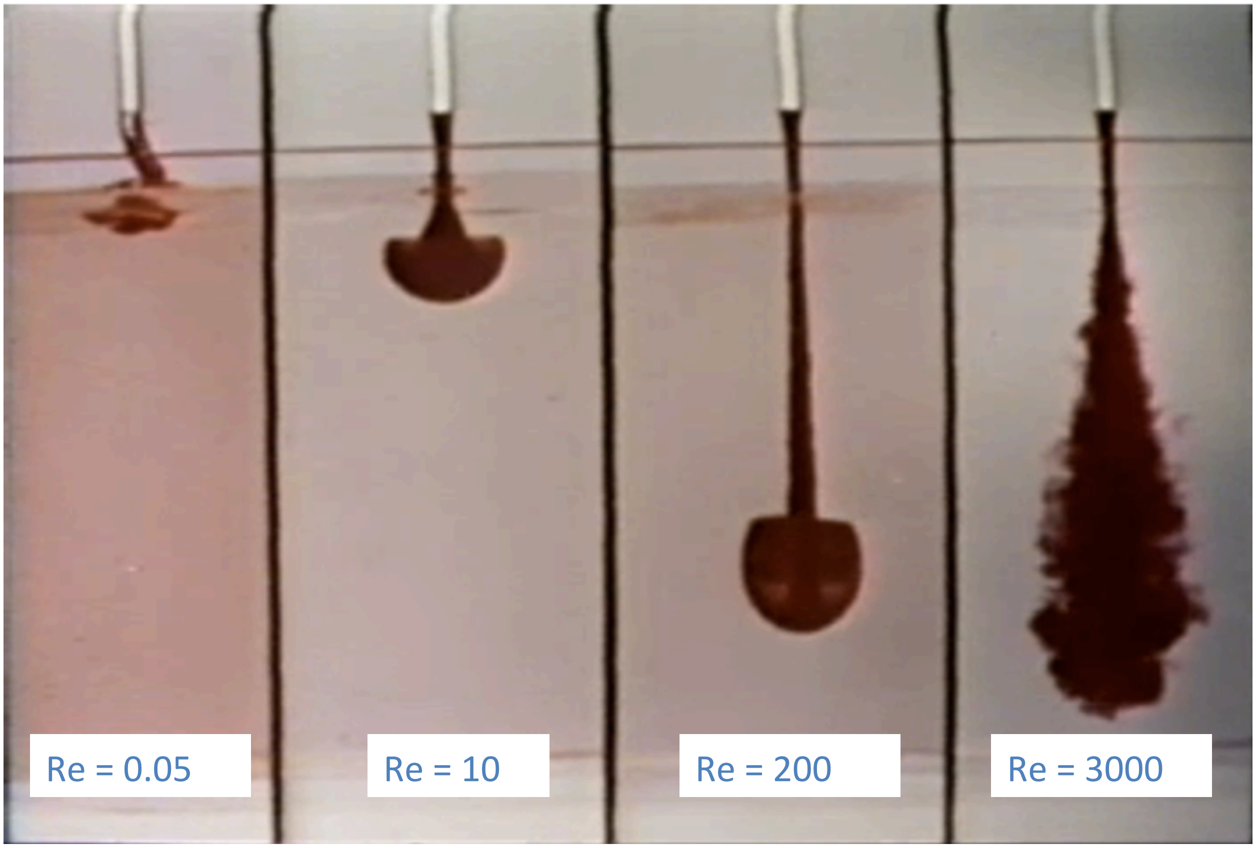

Figure 1: Flows at varying Reynolds number Re. In each panel, a fluid that has been dyed red is injected from the top into the clear fluid on the bottom. The fluids are a glycerin-water mixture, for which the viscosity can be changed by altering the glycerin to water ratio. By changing the viscosity and the injection speed, it is possible to alter the Reynolds number of the injected flow. The frames show how the flow develops as the Reynolds number is varied. This image is a still from the National Committee for Fluid Mechanics Film Series, which, once you get past the distinctly 1960s production values, are a wonderful resource for everything related to fluids.

If we consider the hot phase, which tends to be found around rather than in galactic discs, we reasonable length and velocity scales might be $L \sim 1$ kpc, $V \sim 1000$ km s^{-1}, and if we take $T = 10^7$ K and $n = 10^{-3}$ cm^{-3}, we obtain Re $\sim 10^4$. For the warm neutral or ionised phases in a galactic disc, we might have $L \sim 100$ pc, $V \sim 10$ km s^{-1}, $T \sim 10^4$ K, and $n \sim 1$ cm^{-3}, which gives Re $\sim 10^5$. For a molecular cloud, $L \sim 10$ pc, $V \sim 10$ km s^{-1}, $T \sim 10$ K, and $n \sim 100$ cm^{-3}, giving Re $\sim 10^8$.

We therefore conclude that for all phases of the ISM Re is a very large number. The extremely large value of the Reynolds number immediately yields a critical conclusion: the ISM must be highly turbulent, because flows with Re of more than $\sim 10^3 - 10^4$ invariably are. This is obvious from experiments, as shown in Figure 1.

C. Magnetised flows

We now relax the approximation that there are no magnetic fields present, and generalise our treatment of flows to the magnetic case.

1. Conservation equations

To start with, we will write down the conservation equations for magnetised flows. Conservation of mass is of course the same, but conservation of momentum acquires an extra term representing the forces exerted by magnetic fields on the gas:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v}\mathbf{v}) - \nabla P + \rho\nu\nabla^2\mathbf{v} + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{21}$$

Here $\mathbf{B}$ is the magnetic field.

In addition to the momentum equation, we require an equation to tell us how the magnetic field itself evolves. This is known as the induction equation, and takes the form (again, we assert without proving)

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) = -\nabla \times (\eta \nabla \times \mathbf{B}), \tag{22}$$

where η is the resistivity, which is simply the analog of resistance for a bulk material: if we make a resistor of cross sectional area A and length ℓ out of a material with resistivity η, the resistance will be $\eta\ell/A$.

2. The Alfvén Mach Number

The Mach Number and Reynolds Number for non-magnetised fluids have fairly straightforward analogs for magnetised fluids. We will start with the Mach number. Beginning from the momentum equation, we can make the same order of magnitude estimates of the sizes of terms we did to derive the Reynolds Number, whereby we let L be the characteristic size of the system and V be the characteristic velocity, so L/V is the characteristic timescale. We let B be the characteristic magnetic field strength. Doing so, the order of the various terms in the momentum conservation equation are

$$\frac{\rho V^2}{L} \sim -\frac{\rho V^2}{L} + \frac{\rho c_s^2}{L} + \frac{\rho\nu V}{L^2} + \frac{B^2}{L} \tag{23}$$

$$1 \sim 1 + \frac{c_s^2}{V^2} + \frac{\nu}{VL} + \frac{B^2}{\rho V^2} \tag{24}$$

The second and third terms on the right hand side we have already defined in terms of $\mathcal{M} = V/c_s$ and $\mathrm{Re} = LV/\nu$, and we now see that there is another dimensionless number that characterises the importance of the magnetic term. We define

$$\mathcal{M}_A \equiv \frac{V}{v_A}, \tag{25}$$

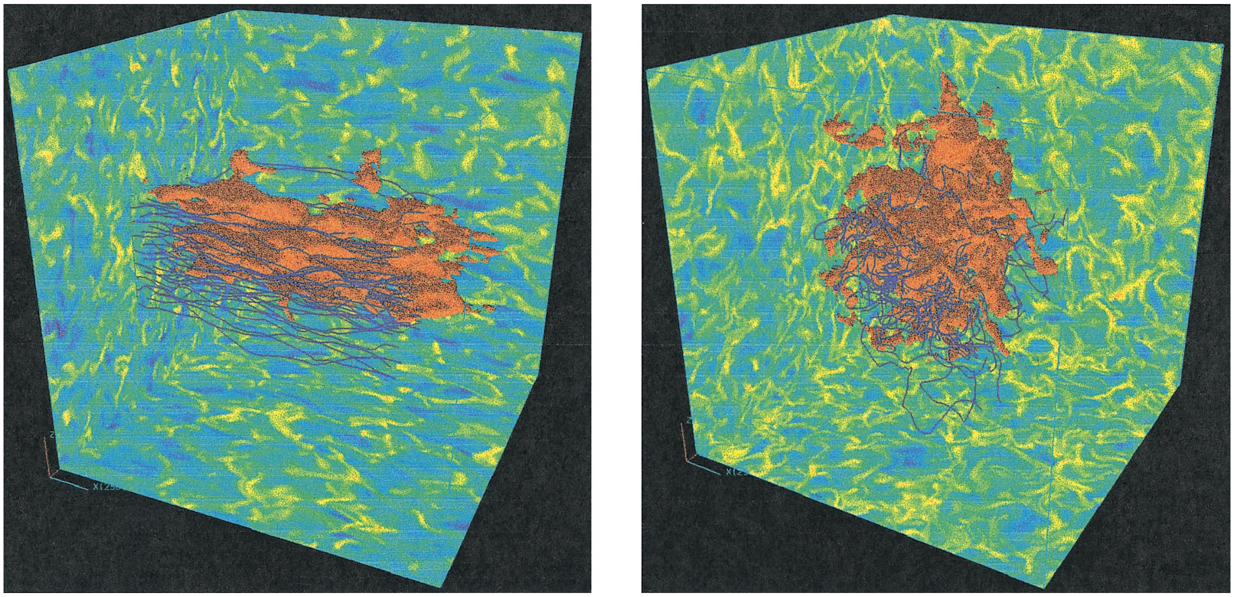

Figure 2: Simulations of sub-Alfvénic (left) and Alfvénic (right) turbulence. Colors on the cube surface are slices of the logarithm of density, blue lines are magnetic field lines, and red surfaces are isodensity surfaces for a passive contaminant added to the flow. From Stone, Ostriker, & Gammie (1998, ApJL, 508, L99).

where

$$v_A = \frac{B}{\sqrt{4\pi\rho}} \tag{26}$$

is the Alfvén speed – the speed of the wave that, in magnetohydrodynamics, plays a role somewhat analogous to the sound wave in hydrodynamics. In flows with $\mathcal{M}_A \gg 1$, which we refer to as super-Alfvénic, the magnetic force term is unimportant, while in those with $\mathcal{M}_A \ll 1$, referred to as sub-Alfvénic, it is dominant.

An important difference between super- and sub-Alfvénic flows is the shape of the magnetic field. Magnetic field lines resist compression via magnetic pressure, and resist bending via magnetic tension; only if magnetic field lines are straight and uniformly spaced do magnetic forces vanish. In the regime $\mathcal{M}_A \ll 1$, these magnetic forces are larger than other forces in the problem, and thus gas thermal pressure and ram pressure are insufficient to bend or compress field lines. As a result, flows with $\mathcal{M}_A \ll 1$ are characterised by having nearly straight, uniform magnetic fields. Conversely, flows with $\mathcal{M}_A \gg 1$ are characterised by having bend, tangled magnetic fields.

3. The magnetic Reynolds Number

   Now let us repeat this scaling exercise for the induction equation, equation 22. The various terms scale as

$$\frac{BV}{L} + \frac{BV}{L} \quad \sim \quad \eta \frac{B}{L^2} \tag{27}$$

$$1 \sim \frac{\eta}{VL} \tag{28}$$

Note that the scaling of the final term looks very much like the one we obtained for the Reynolds Number for hydrodynamics, i.e., it is resistivity / viscosity times V / B divided by L^2. In analogy to the ordinary hydrodynamic Reynolds number, we therefore define the magnetic Reynolds number by

$$\mathrm{Rm} = \frac{LV}{\eta}. \tag{29}$$

The significance of η, and the magnetic Reynolds Number, becomes clear if we investigate the implications of the limit $\eta \to 0$, or $\mathrm{Rm} \to \infty$. If we adopt $\eta = 0$ exactly, we have

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) = 0. \tag{30}$$

To understand what this equation implies, it is useful consider the magnetic flux Φ threading some fluid element. We define this as

$$\Phi = \int_A \mathbf{B} \cdot \hat{\mathbf{n}} \, dA, \tag{31}$$

where we integrate over some area A that defines the fluid element. The time derivative of this is then

$$\begin{aligned} \frac{d\Phi}{dt} &= \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \, dA + \oint_{\partial A} \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l} && (32) \\ &= \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \, dA + \oint_{\partial A} \mathbf{B} \times \mathbf{v} \cdot d\mathbf{l} && (33) \end{aligned}$$

where ∂A is the boundary of A. Here the second term on the right comes from the fact that, if the fluid is moving at velocity $\mathbf{v}$, the area swept out by a vector $d\mathbf{l}$ per unit time is $\mathbf{v} \times d\mathbf{l}$, so the flux crossing this area is $\mathbf{B} \cdot \mathbf{v} \times d\mathbf{l}$. Then in the second step we used the fact that $\nabla \cdot \mathbf{B} = 0$ to exchange the dot and cross products.

If we now apply Stokes theorem again to the second term, we get

$$\begin{aligned} \frac{d\Phi}{dt} &= \int_A \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} \, dA + \int_A \nabla \times (\mathbf{B} \times \mathbf{v}) \cdot \hat{\mathbf{n}} \, dA && (34) \\ &= \int_A \left[\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}) \right] \cdot \hat{\mathbf{n}} \, dA && (35) \\ &= 0. && (36) \end{aligned}$$

Thus the magnetic flux through each fluid element is a conserved quantity. A useful analogy is that fluid elements are tied to magnetic field lines like beads on a wire; they are free to slide up and down the wire, and encounter no resistance when they do so, but they can never come off the wire.

We call the regime $\mathrm{Rm} \gg 0$, where this condition of flux freezing holds, ideal magnetohydrodynamics. Flows where $\mathrm{Rm} \lesssim 1$ are non-ideal MHD. Note, however, that like viscosity Rm depends on length scale, so for sufficiently small L, we will always reach a regime where $\mathrm{Rm} \lesssim 1$ even if $\mathrm{Rm} \gg 1$ on larger scales. This is a size scale on which magnetic fields and matter are no longer perfectly flux-frozen to one another, and is of order $L \sim (\eta/V)\mathrm{Rm}$.

4. Non-ideal MHD effects and magnetic Reynolds Numbers in the ISM

Effects that make η non-zero, and allow violation of flux freezing, are called non-ideal MHD effects. There are a wide range of non-ideal effects, only some of which are important for ISM conditions. In the parts of the ISM that are fully ionised, the main effect is collisions between ions and electrons in a highly-ionised plasma. In an ideal plasma electrons and ions do not collide, and interact only via electric forces, which cause currents to flow. Collisions impede the flow of current, thereby giving rise to resistivity. This produces a resistivity, called Spitzer resistivity,

$$\eta \approx \frac{4\sqrt{2\pi}}{3} \frac{Ze^2 m_e^{1/2} \ln\Lambda}{(k_B T_e)^{3/2}}, \tag{37}$$

where Z is the mean ion charge, T_e is the electron temperature, and $\ln\Lambda$ is the Coulomb logarithm, a number ≈ 10 that accounts for shielding of the plasma of large distances by electrons clustering around ions. If we consider most of the ISM, where the dominant ion is hydrogen, $Z = 1$, then we roughly have

$$\eta \approx 10^{-13} \left(\frac{T_e}{10^4\ \mathrm{K}}\right)^{-3/2} \mathrm{cm}^2\ \mathrm{s}^{-1} \tag{38}$$

We can immediately see from this expression that the ionised parts of the ISM are very close to the ideal MHD regime, since $\mathrm{Rm} \sim LV/\eta$. On interstellar scales $LV \gg 1\ \mathrm{cm}^2\ \mathrm{s}^{-1}$, while we have just shown that $\eta \ll 1\ \mathrm{cm}^2\ \mathrm{s}^{-1}$, so $\mathrm{Rm} \gg 1$ at all times. Flux freezing is a *very* good approximation for the ionised parts of the ISM.

In the part of the ISM that are predominantly neutral, but where there are still some ions, the main non-ideal effect is called ion-neutral drift, or ambipolar diffusion. We can understand this effect intuitively as follows. Only ions and electrons feel the Lorentz force exerted by a magnetic field directly; neutrals do not. This means that magnetic fields only exert forces on neutral particles indirectly, by exerting forces on the ions and electrons (and mostly ions matter for this purpose), which then collide with the neutrals, exchanging momentum with them and therefore transmitting the magnetic force. However, if the ionisation fraction is sufficiently small, then a neutral atom may have to wait a long, long time before it has a collision with an ion; there are just very few ions with which to collide.

Because neutrals encounter ions only rarely, an appreciable difference can build up between the mean velocity of the ions, which feel magnetic forces, and the neutrals, which do not. To estimate how this process works, we need to think about the forces acting on both ions and neutrals. The ions feel a Lorentz force

$$\mathbf{f}_L = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{39}$$

The other force in play is the drag force due to ion-neutral collisions, which is

$$\mathbf{f}_d = \gamma \rho_n \rho_i (\mathbf{v}_i - \mathbf{v}_n), \tag{40}$$

where the subscript i and n refer to ions and neutrals, respectively, and γ is the drag coefficient, which can be computed from the microphysics of the plasma. In a very weakly ionised fluid, the neutrals and ions very quickly reach terminal velocity with respect to one another, so the drag force and the Lorentz force must balance. Equating our expressions and solving for $\mathbf{v}_d = \mathbf{v}_i - \mathbf{v}_n$, the drift velocity, we get

$$\mathbf{v}_d = \frac{1}{4\pi\gamma\rho_n\rho_i}(\nabla \times \mathbf{B}) \times \mathbf{B} \tag{41}$$

To figure out how this affects the fluid, we write down the equation of magnetic field evolution under the assumption that the field is perfectly frozen into the ions:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}_i) = 0. \tag{42}$$

To figure out how the field behaves with respect to the neutrals, which constitute most of the mass, we simply use our expression for the drift speed $\mathbf{v}_d$ to eliminate $\mathbf{v}_i$. With a little algebra, the result is

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{v}_n) = \nabla \times \left\{ \frac{\mathbf{B}}{4\pi\gamma\rho_n\rho_i} \times [\mathbf{B} \times (\nabla \times \mathbf{B})] \right\}. \tag{43}$$

Referring back to the induction equation (22), we can see that the resistivity produced by ion-neutral drift is not a scalar, and that it is non-linear, in the sense that it depends on $\mathbf{B}$ itself.

However, our scaling analysis still applies. The magnitude of the resistivity produced by ion-neutral drift is

$$\eta_{\mathrm{AD}} = \frac{B^2}{4\pi\rho_i\rho_n\gamma}. \tag{44}$$

Thus, the magnetic Reynolds number is

$$\mathrm{Rm} = \frac{LV}{\eta_{\mathrm{AD}}} = \frac{4\pi LV\rho_i\rho_n\gamma}{B^2} \approx \frac{4\pi LV\rho^2 x\gamma}{B^2}, \tag{45}$$

where $x = n_i/n_n$ is the ion fraction, which we've assumed is $\ll 1$ in the last step. Ion-neutral drift will allow the magnetic field lines to drift through the fluid on length scales L such that $\mathrm{Rm} \lesssim 1$. Thus, we can define a characteristic length scale for ambipolar diffusion by

$$L_{\mathrm{AD}} = \frac{B^2}{4\pi\rho^2 x\gamma V}. \tag{46}$$

In order to evaluate this numerically, we must calculate the ion-neutral drag coefficient γ. The dominant effect at low speeds is that ions induce a dipole moment in nearby neutrals, which allows them to undergo a Coulomb interaction. This greatly enhances the cross-section relative to the geometric value. We will not go into details of that calculation, and will simply adopt the results: $\gamma \approx 9.2 \times 10^{13}$ cm^3 s^{-1} g^{-1} (Smith & Mac Low, 1997, A&A, 326, 801). Thus we have

$$\mathrm{Rm} \sim 20 \left(\frac{L}{\mathrm{pc}}\right) \left(\frac{V}{\mathrm{km\ s^{-1}}}\right) \left(\frac{n}{1\ \mathrm{cm}^3}\right)^2 \left(\frac{x}{10^{-2}}\right) \left(\frac{B}{\mu\mathrm{G}}\right)^{-2}, \tag{47}$$

where n is the number density of H nuclei. Thus we see that the atomic ISM, which tends to have $x \sim 10^{-2}$, $L \sim 10 - 100$ pc, and $n \sim 1$ cm^{-3} is safely in the ideal MHD regime, $\mathrm{Rm} \gg 1$. The situation is less clear for the molecular ISM, which is much more weakly ionised, $x \sim 10^{-6}$, and tends to have much smaller scale structures and stronger magnetic fields. In the molecular ISM, non-ideal MHD effects can become significant on small scales.