Class 7: Gravitational instability and collapse

ASTR 4008 / 8008, Semester 2, 2020

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Outline

- The virial theorem (keynote keeps trying to autocorrect this to “viral” — not helpful right now!)
- “Virial” competition
  - Gravity vs. thermal support: the Jeans instability
  - Gravity vs. magnetic support: the magnetic critical mass
  - Gravity vs. turbulence: the virial parameter
- Pressureless collapse

The virial theorem

What is it and why use it

- The virial theorem is a volume-integrated version of the equations of motion
- It can be used to describe the overall expansion or contraction of a volume — we will define what we mean by this more precisely as we proceed
- From our standpoint it is mostly a tool to understand which forces promote collapse and which forces oppose it, and to get rough estimates under what circumstances those forces should prevail

Virial theorem

Derivation I

- Start from equations of mass and momentum conservation, omitting dissipative terms (viscosity, resistivity) since they are small on large scales:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v}\mathbf{v}) - \overset{\text{Pressure}}{\nabla P} + \overset{\text{Lorentz force}}{\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}} - \overset{\text{Gravity}}{\rho \nabla \phi}$$

Gravitational potential

- First step: rewrite in manifestly tensorial form:

Reynolds stress tensor

$$\mathbf{\Pi} \equiv \rho \mathbf{v}\mathbf{v} + P\mathbf{I}$$

Identity tensor

$$\mathbf{T}_M \equiv \frac{1}{4\pi}\left(\mathbf{B}\mathbf{B} - \frac{B^2}{2}\mathbf{I}\right)$$

Maxwell stress tensor

$$\longrightarrow \quad \frac{\partial}{\partial t}(\rho \mathbf{v}) = \nabla \cdot (\mathbf{\Pi} - \mathbf{T}_M) - \rho \nabla \phi$$

Virial theorem

Derivation II

- Define arbitrary fixed volume *V*, define moment of inertia $I = \int_V \rho r^2 \, dV$
- Compute rate of change of *I*:

$$\dot{I} = \int_V \frac{\partial \rho}{\partial t} r^2 \, dV$$

V not time-variable so take time derivative inside integral

$$= -\int_V \nabla \cdot (\rho \mathbf{v}) r^2 \, dV$$

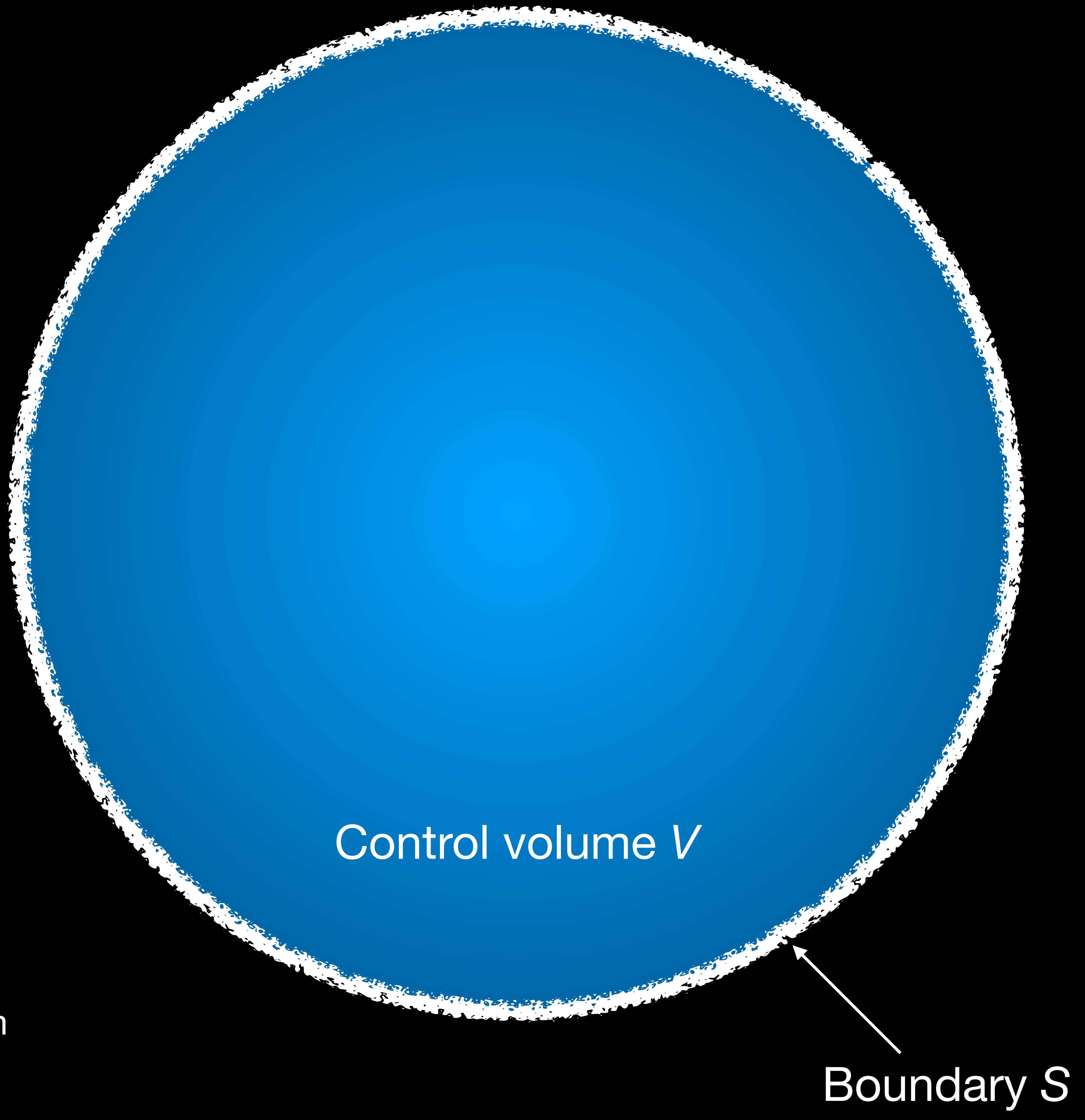
Use mass conservation

$$= -\int_V \nabla \cdot (\rho \mathbf{v} r^2) \, dV + 2 \int_V \rho \mathbf{v} \cdot \mathbf{r} \, dV$$

Bring r^2 factor inside divergence

$$= -\int_S (\rho \mathbf{v} r^2) \, d\mathbf{S} + 2 \int_V \rho \mathbf{v} \cdot \mathbf{r} \, dV$$

Use divergence theorem on first term



Virial theorem

Derivation III

- Now take time derivative a second time

$$\ddot{I} = -\frac{d}{dt}\int_S r^2(\rho\mathbf{v})\cdot d\mathbf{S} + \int_V \frac{\partial}{\partial t}(\rho\mathbf{v})\cdot\mathbf{r}\,dV$$

Take time derivative inside integral in second term

$$= -\frac{d}{dt}\int_S r^2(\rho\mathbf{v})\cdot d\mathbf{S} - \int_V \mathbf{r}\cdot\left[\nabla\cdot(\mathbf{\Pi}-\mathbf{T}_M)+\rho\nabla\phi\right]\,dV$$

Use momentum conservation equation

- Next prove a simple tensor identity (tensor analog to divergence theorem):

$$\int_V \mathbf{r}\cdot\nabla\cdot\mathbf{T}\,dV = \int_V x_i\frac{\partial}{\partial x_j}T_{ij}\,dV$$

Rewrite in index notation (for convenience)

$$= \int_V \frac{\partial}{\partial x_j}\left(x_i T_{ij}\right)\,dV - \int_V T_{ij}\frac{\partial}{\partial x_j}x_i\,dV$$

Bring x_i inside derivative

$$= \int_S x_i T_{ij}\,dS_j - \int_V T_{ij}\delta_{ij},dV$$

Apply divergence theorem to first term, orthogonality of unit vectors in second term

$$= \int_S \mathbf{r}\cdot\mathbf{T}\cdot d\mathbf{S} - \int_V \mathrm{Tr}\mathbf{T}\,dV$$

Rewrite in vector notation

Trace = sum of diagonal elements

Virial theorem

Derivation IV

- Use identity to evaluate divergence, noting Tr $\mathbf{\Pi} = 3P + \rho v^2$, Tr $\mathbf{T}_M = -B^2 / 8\pi$:

$$\frac{1}{2}\ddot{I} = 2\left(\mathcal{T} - \mathcal{T}_S\right) + \mathcal{B} + \mathcal{W} - \frac{1}{2}\frac{d}{dt}\int_S \left(\rho \mathbf{v} r^2\right) \cdot d\mathbf{S}$$

- Terms appearing here:

Kinetic + thermal energy

$$\mathcal{T} = \int_V \left(\frac{1}{2}\rho v^2 + \frac{3}{2}P\right) dV$$

Magnetic energy + magnetic stress at surface

$$\mathcal{B} = \frac{1}{8\pi}\int_V B^2\, dV + \int_S \mathbf{r} \cdot \mathbf{T}_M \cdot d\mathbf{S}$$

Terms that generally oppose collapse (positive terms)

$$\mathcal{T}_S = \int_S \mathbf{r} \cdot \mathbf{\Pi} \cdot d\mathbf{S}$$

Fluid pressure / stress at surface

$$\mathcal{W} = -\int_V \rho \mathbf{r} \cdot \nabla\phi\, dV$$

Gravitational potential energy

Terms that generally promote collapse (negative terms)

Change due to advection across surface

$$-\frac{1}{2}\frac{d}{dt}\int_S \left(\rho \mathbf{v} r^2\right) \cdot d\mathbf{S}$$

Terms that can have either sign

Thermal pressure versus gravity

Jeans analysis

- Most basic force opposing collapse is pressure
- Consider spherical cloud of mass *M*, radius *R*, with constant sound speed c_s; virial theorem terms are

Constant of order unity, depends on density profile; for ρ = const, *a* = 3/5

$$\mathcal{T} = \int_V \frac{3}{2} P \, dV = \frac{3}{2} \int_V \rho c_s^2 \, dV = \frac{3}{2} M c_s^2 \qquad \mathcal{W} = -a \frac{GM^2}{R}$$

- Gravity should win if $R \lesssim \frac{GM}{c_s^2}$ or equivalently $R \gtrsim \frac{c_s}{\sqrt{G\rho}}$
- Physical interpretation: for a fixed cloud mass, if cloud gets too compressed, gravity wins and collapse likely; equivalently, for a fixed gas density, if region is too large, gravity wins

Jeans stability analysis

Part I

- Consider a uniform, infinite, isothermal medium; governing equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Mass conservation

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla P - \rho \nabla \phi$$

Momentum conservation

$$\nabla^2 \phi = 4\pi G \rho$$

Poisson equation

- "Jeans swindle": this isn't really a proper background state, because potential is undefined for a uniform, infinite medium, but ignore that…

- Consider a small perturbation on this: $\rho = \rho_0 + \epsilon\rho_1$, $v = \epsilon v_1$, $\phi = \phi_0 + \epsilon\phi_1$, $\epsilon \ll 1$

- Treat perturbations as a Fourier mode: $\rho_1 = \rho_a \exp[\, i\,(kx - \omega t)\,]$

Jeans stability analysis

Part II

- Substitute perturbation into Poisson equation: $\nabla^2(\phi_0 + \epsilon\phi_1) = 4\pi G(\rho_0 + \epsilon\rho_1)$
- Parts involving ρ_0, ϕ_0 cancel because they are solution to unperturbed eqn; remaining part is $\nabla^2\phi_1 = 4\pi G\rho_a e^{i(kx-\omega t)} \implies \phi_1 = \frac{4\pi G\rho_a}{k^2}e^{i(kx-\omega t)}$
- Next repeat process for mass conservation equation:

$$\frac{\partial}{\partial t}(\rho_0 + \epsilon\rho_1) + \nabla \cdot [(\rho_0 + \epsilon\rho_1)(\epsilon v_1)] = 0$$

Substitute in

$$\frac{\partial}{\partial t}\rho_0 + \epsilon\frac{\partial}{\partial t}\rho_1 + \epsilon\nabla \cdot (\rho_0 v_1) = 0$$

Drop terms of order ϵ^2

$$\frac{\partial}{\partial t}\rho_1 + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0$$

Background density
ρ_0 = constant

- This is called the linearised equation

Exercise: obtain the linearised momentum equation

Jeans stability analysis

Part III

- Linearised momentum equation: $\frac{\partial}{\partial t}\left(\rho_0 \mathbf{v_1}\right) = -c_s^2 \nabla \rho_1 - \rho_0 \nabla \phi_1$
- Substitute Fourier modes into mass conservation equation:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\rho_a e^{i(kx-\omega t)}\right) + \nabla \cdot \left(\rho_0 \mathbf{v}_a e^{i(kx-\omega t)}\right) &= 0 \\ -i\omega \rho_a e^{i(kx-\omega t)} + ik\rho_0 v_{a,x} e^{i(kx-\omega t)} &= 0 \\ -\omega \rho_a + k\rho_0 v_{a,x} &= 0 \\ \frac{\omega \rho_a}{k\rho_0} &= v_{a,x} \end{aligned}$$

- Same process for linearised momentum equation: $\omega \rho_0 v_{a,x} = k\left(c_s^2 \rho_a + \rho_0 \phi_a\right)$

Jeans stability analysis

Jeans length and Jeans mass

- Substitute ϕ_a and $v_{a,x}$ into linearised momentum equation $\omega \rho_0 v_{a,x} = k\left(c_s^2 \rho_a + \rho_0 \phi_a\right)$
- Result is a dispersion relation: $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$
- Critical value of $k = k_J \equiv \sqrt{4\pi G \rho_0 / c_s^2}$
  - $k > k_J \rightarrow \omega$ real, so amplitude of perturbation constant, varying phase
  - $k < k_J \rightarrow \omega$ imaginary, amplitude of perturbation grows exponentially
- Jeans wavelength $\lambda_J = 2\pi/k_J = c_s\ (\pi\ /\ G\rho_0)^{1/2}$, mass $M_J \sim \rho_0\ \lambda_J^3 \sim c_s^3\ /\ (G^3\rho)^{1/2}$
- Growth time of unstable mode $t_{\mathrm{gr}} \sim 1\ /\ |\omega| \sim (G\rho_0)^{-1/2}$

Magnetic pressure vs. gravity

Magnetic critical mass

- Magnetic and gravitational terms in virial theorem:

$$\mathcal{B} = \frac{1}{8\pi}\int_V B^2\,dV + \int_S \mathbf{r}\cdot\mathbf{T}_M\cdot d\mathbf{S} \qquad \mathcal{W} = -\int_V \rho\mathbf{r}\cdot\nabla\phi\,dV$$

- Consider spherical cloud of mass *M*, radius *R*, with uniform field *B*, surface field much weaker than field in cloud, so we can neglect surface term; then

$$\mathcal{B} \approx \frac{B^2R^3}{6} = \frac{\Phi_B^2}{6\pi^2R} \qquad \mathcal{W} = -a\frac{GM^2}{R}$$

Magnetic flux $\Phi_B = \pi R^2 B$; constant under ideal MHD

- Both terms ~ 1/*R*, so relative strength depends only on mass: gravity wins if $M > M_\Phi$, where M_Φ is called the magnetic critical mass,

$$M_\Phi = \sqrt{\frac{5}{2}}\left(\frac{\Phi_B}{3\pi G^{1/2}}\right)$$

Turbulence vs. gravity

The virial parameter

- Uniform spherical cloud with velocity dispersion σ; VT terms are

$$\mathcal{T} = \int_V \frac{1}{2}\rho v^2 \, dV = \frac{1}{2}M\sigma^2 \qquad \mathcal{W} = -a\frac{GM^2}{R}$$

- Define the virial ratio: $\alpha_{\rm vir} = \dfrac{5\sigma^2 R}{GM}$
- Gravity wins if $\alpha_{\rm vir} < 1$
- However, even if turbulence inhibits *global* collapse, it does not prevent *local* collapse, in places where velocity field is converging — in terms of the VT, this shows up in the surface term $\mathscr{T}_{\rm s}$

Pressureless collapse

Basic considerations

- In any place where gravity wins, it tends to “run away” due to the $1/R$ dependence in the virial theorem — it wins a little at first, but then becomes increasingly dominant as collapse proceeds
- This motivates exploration of the limiting case where there are no significant forces opposing gravity: pressureless collapse
- This has the advantage that it can easily be solved analytically; analytic solutions exist for some other cases too (as you will show in your homework), but this is the most straightforward

Pressureless collapse

Shell dynamics

- Work in terms of mass shells; let $M(r)$ = mass interior to r, dM/dr = $4\pi\, r^2\rho$
- Equation of mass conservation for mass shells:

$$
\begin{aligned}
\frac{\partial M}{\partial t} &= 4\pi \int_0^r r'^2 \frac{\partial \rho}{\partial t}\, dr' && \text{Take derivative inside integral} \\
&= -4\pi \int_0^r r'^2 \nabla \cdot (\rho \mathbf{v})\, dr' && \text{Use equation of mass conservation} \\
&= -4\pi \int_0^r \frac{\partial}{\partial r'} (r'^2 \rho v)\, dr' && \text{Write out divergence in spherical symmetry} \\
&= -4\pi r^2 \rho v && \text{Fundamental theorem of calculus} \\
&= -v \frac{\partial M}{\partial r} && \text{Definition of } M
\end{aligned}
$$

Radial velocity

Pressureless collapse

The free-fall time

Lagrangian derivative (taken following a particular shell)

- Momentum equation in Lagrangian form: $\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial r} - \rho \frac{GM}{r^2}$
- Consider collapse with zero pressure starting from rest at $r = r_0$; solution is

$$v = \frac{dr}{dt} = -\sqrt{2GM} \left(\frac{1}{r} - \frac{1}{r_0} \right)^{1/2} \quad \Rightarrow \quad \frac{dr}{\sqrt{r_0/r - 1}} = -\sqrt{2GM/r_0}\, dt$$

- Integrate again (done by trig substitution): $t = \sqrt{\frac{r_0^3}{2GM}} \left(\xi + \frac{1}{2} \sin 2\xi \right), \quad \frac{r}{r_0} = \cos^2 \xi$
- Shell reaches $r = 0$ (corresponds to $\xi = \pi/2$) at

$$t = t_{\text{ff}} \equiv \frac{\pi}{2} \sqrt{\frac{r_0^3}{2GM}} = \sqrt{\frac{3\pi}{32 G\rho}}$$

Free-fall time

Mean density of material interior to r_0 at $t = 0$

Implications of pressureless collapse

- Maximum mass of pressure-supported object is ~ $M_J \sim \rho_0 \lambda_J^3 \sim c_s^3 / (G^3 \rho)^{1/2}$, so if collapse starts from such an object, mean accretion rate $(dM/dt) \sim M_J / t_{\rm ff} \sim c_s^3 / G \sim 10^{-6}\ M_\odot\ \rm yr^{-1}$ at $T = 10$ K, independent of density!

- Time to reach origin depends only on density interior to starting radius:
  - Uniform-density clouds collapse all at once
  - Centrally concentrated clouds collapse “inside-out”: density $\rho = \rho_c\ (r / r_c)^{-\alpha}$ gives collapse time for shell starting at r_0 of $t(r_0) = t_{\rm ff}(\rho_c)\ (r_0 / r_c)^{\alpha/2}$

- Once a given shell reaches $r \ll r_0$, $v \approx v_{\rm ff} \equiv (2GM / r)^{1/2}$

- Density profile near star: $\frac{\partial M}{\partial t} = -\frac{\partial M}{\partial r} = -4\pi r^2 \rho v \quad \Longrightarrow \quad \rho = \frac{\dot{M}}{4\pi\sqrt{2GM}} r^{-3/2}$