Class 5: Gas flows and turbulence

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Outline

- Fundamental numbers: the Reynolds number and the Mach number
- Statistics of turbulence
  - Power spectra and autocorrelation functions
  - The Kolmogorov model
- Supersonic turbulence
  - Velocity statistics
  - Density statistics

Fluid dynamics

A *very* condensed introduction

- Basic equations relevant for us are conservation of mass and momentum:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \qquad \frac{\partial}{\partial t}(\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla P + \rho \nu \nabla^2 \mathbf{v}$$

Density — Velocity — Momentum density — Tensor product — Pressure — Viscosity

- Momentum equation in index notation:

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial}{\partial x_j}(\rho v_i v_j) - \frac{\partial}{\partial x_i} P + \rho \nu \frac{\partial^2}{\partial x_j^2} v_i$$

- Key role of viscosity: viscous term is the only one that dissipates into heat

Characteristic scales

Really all we know how to do in fluids…

- Suppose we consider a system of characteristic size scale L, velocity scale V, sound speed c_s; natural time scale is $T = L / V$
- How big (order of magnitude) are terms in momentum equation:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla P + \rho \nu \nabla^2 \mathbf{v}$$

- Answer:

$$\frac{\rho V^2}{L} \sim \frac{\rho V^2}{L} + \frac{\rho c_s^2}{L} + \rho \nu \frac{V}{L^2} \quad \rightarrow \quad 1 \sim 1 + \frac{c_s^2}{V^2} + \frac{\nu}{VL}$$

↑ ↑

Define $V / c_s = \mathscr{M}$, Mach number

Define $LV / \nu = \mathrm{Re}$, Reynolds number

What are $\mathscr{M}$ and Re?

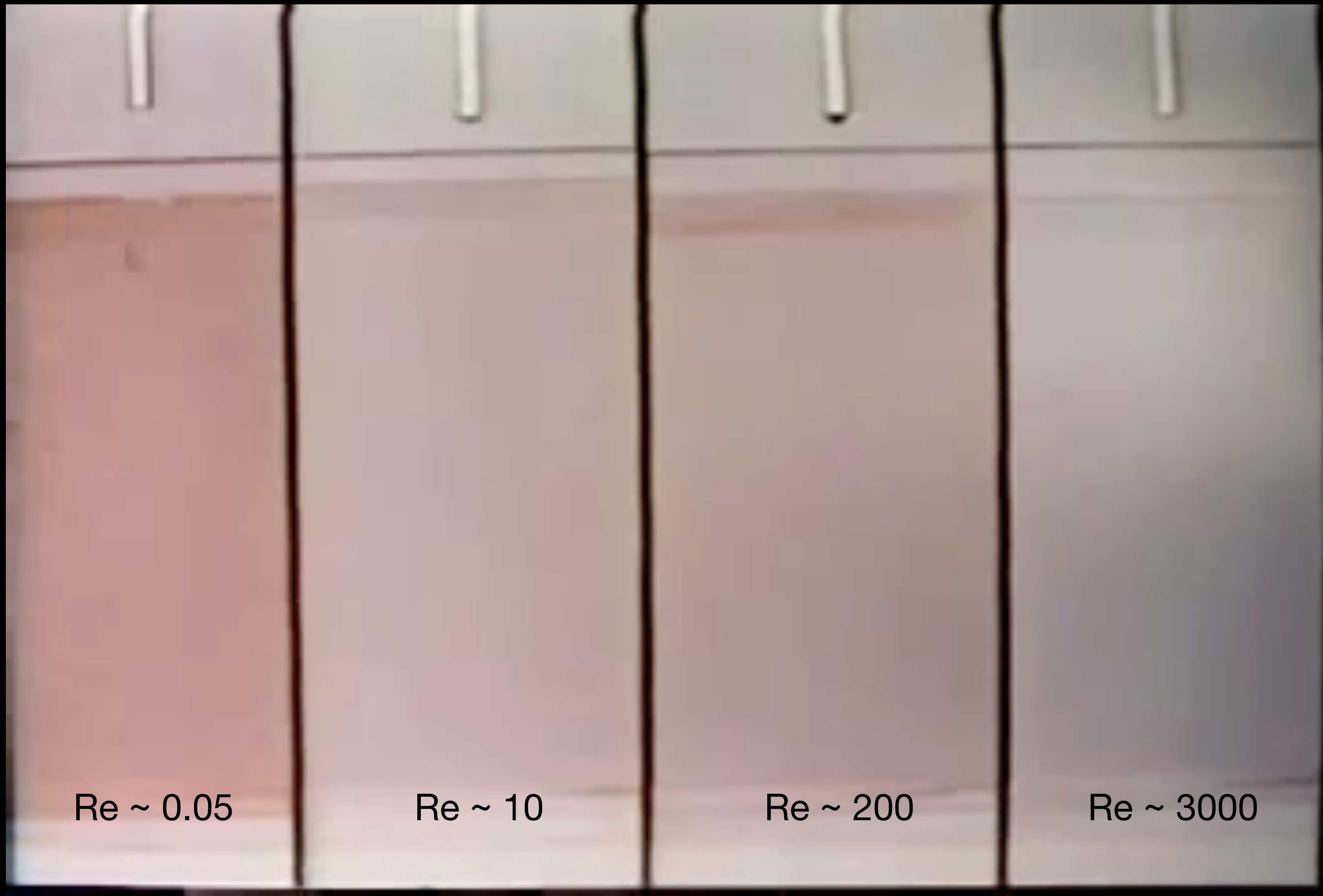
Typical molecular cloud values

- We saw earlier that typical length, speed values are L ~ 10 pc, V ~ few km s^{-1}
- Sound speed $c_s = (kT / \mu m_H)^{1/2} \approx 0.2$ km s^{-1} at T = 10 K, $\mu = 2.33 \Rightarrow \mathscr{M} \sim 10$
- Viscosity of ideal gas $\nu = 2u_{rms}\lambda_{mfp}$, u_{rms} = RMS particle speed $\approx c_s$, λ_{mfp} = mean free path = $1 / n\sigma$, where n = number density ~ 100 cm^{-3}, σ = cross section, typically ~(1 nm)^2 for neutral molecules $\Rightarrow \nu \sim 10^{16}$ cm^2 s^{-1}, Re ~ 10^9
- Thus molecular clouds are highly supersonic, very high Re $\Rightarrow$ pressure and viscous forces largely unimportant

Why Re matters

High Re flows are inevitably turbulent

NSF Fluid Mechanics Film Series, https://www.youtube.com/playlist?list=PL0EC6527BE871ABA3



Molecular clouds, Re ~ 10^9

Exercise for class: estimate the Reynolds number of the air in this room. Do you expect air flows in this room to be turbulent?

Turbulence statistics

Velocities

- Let **v**(**x**) be the velocity at position **x** in some volume of interest *V*
- Define the auto-correlation function (ACF): $A(\mathbf{r}) \equiv \frac{1}{V} \int \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}+\mathbf{r})\, d\mathbf{x}$
- Fourier space equivalent: power spectral density (PSD): $\Psi(\mathbf{k}) = |\tilde{\mathbf{v}}(\mathbf{k})|^2$, where

$$\tilde{\mathbf{v}}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \mathbf{v}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}$$

- ACF and PDF both measure how quickly velocity changes with separation; if turbulence is isotropic, ACF depends only on $r = |\mathbf{r}|$; PDF depends on $k = |\mathbf{k}|$
- Define the turbulent power spectrum $P(k) = 4\pi k^2 \Psi(k)$

PSD and ACF

Two sides of the same coin

- PDF and ACF are closely related, and both measure power in turbulent motions on different size scales
- Wiener-Khinchin Theorem: $\Psi(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int A(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}$
- Parseval's theorem: $\int P(k)\, dk = \int |\tilde{\mathbf{v}}(\mathbf{k})|^2\, d\mathbf{k} = \int \mathbf{v}(\mathbf{x})^2\, d\mathbf{x}$
- Physical meaning: ACF and PSD are just each other's Fourier transforms, and integral of power spectrum gives total power, which is the same in either real or Fourier space

The linewidth-size relation

Observable manifestation of power spectra

- We cannot measure the power spectrum directly in the ISM; what we can measure is the linewidth-size relation (LWS): measure the velocity dispersion $\sigma(\ell)$ (linewidth) in regions of varying size ℓ, and plot the correlation
- LWS related to power spectrum: consider a power spectrum $P(k) \sim k^{-n}$
- To compute the LWS, look at KE per unit mass of material in a box of size ℓ: KE ~ $\sigma(\ell)^2$, and also $\mathrm{KE} \sim \int_{2\pi/\ell}^{\infty} P(k)\, dk \propto \ell^{n-1}$
- Therefore $\sigma(\ell) = c_s\, (\ell / \ell_s)^{(n-1)/2}$, where ℓ_s = sonic length

The Kolmogorov model

for subsonic turbulence

- Basic picture: in the absence of shocks, equations of hydrodynamics in Fourier space are *local*, i.e., energy is transferred only between adjacent k values
- Viscosity only important on scales $L_{\rm diss}$ small enough that Re = $L_{\rm diss} V / \nu \sim 1$
- Energy injected at a much larger scales $L_{\rm inj}$
- In between injection and dissipation scale is *inertial range*, $1/L_{\rm inj} \ll k \ll 1/L_{\rm diss}$: in this range of k, energy flows through, but no sources or sinks of energy
- In steady state, power $P(k)$ in inertial range can only depend on k and on rate of dissipation (= rate of injection) ψ

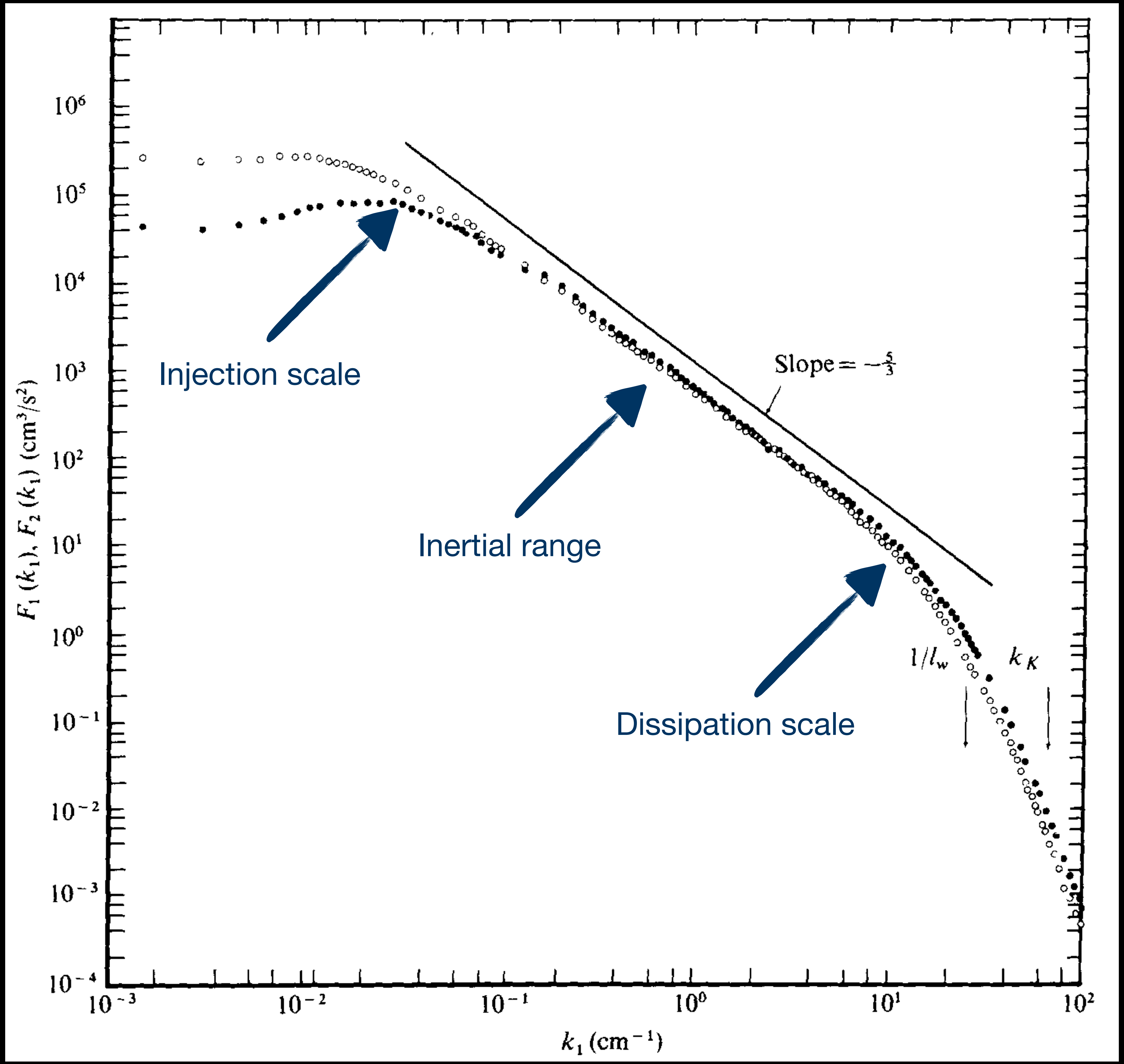
The Kolmogorov scaling

and the magic of dimensional analysis

- Since $P(k)$ depends only on k and ψ, we have $P(k) \propto k^{\alpha} \psi^{\beta}$ for some α, β
- Kolmogorov's argument: k has units of 1/length = $1/L$, ψ has units of energy per unit mass per unit time = L^2 / T^3, and $P(k)$ has units of energy per unit mass per unit k = L^3 / T^2; thus we have $L^3 / T^2 = L^{-\alpha} (L^2 / T^3)^{\beta}$
- For units to match, only solution is $\beta = 2/3$, $\alpha = -5/3$
- Therefore $P(k) \propto k^{-5/3}$, LWS relation $\sigma \propto \ell^{1/3}$

Experimental verification of Kolmogorov model

Test using turbulence from an air jet



Champagne 1978, J. Fluid Mech

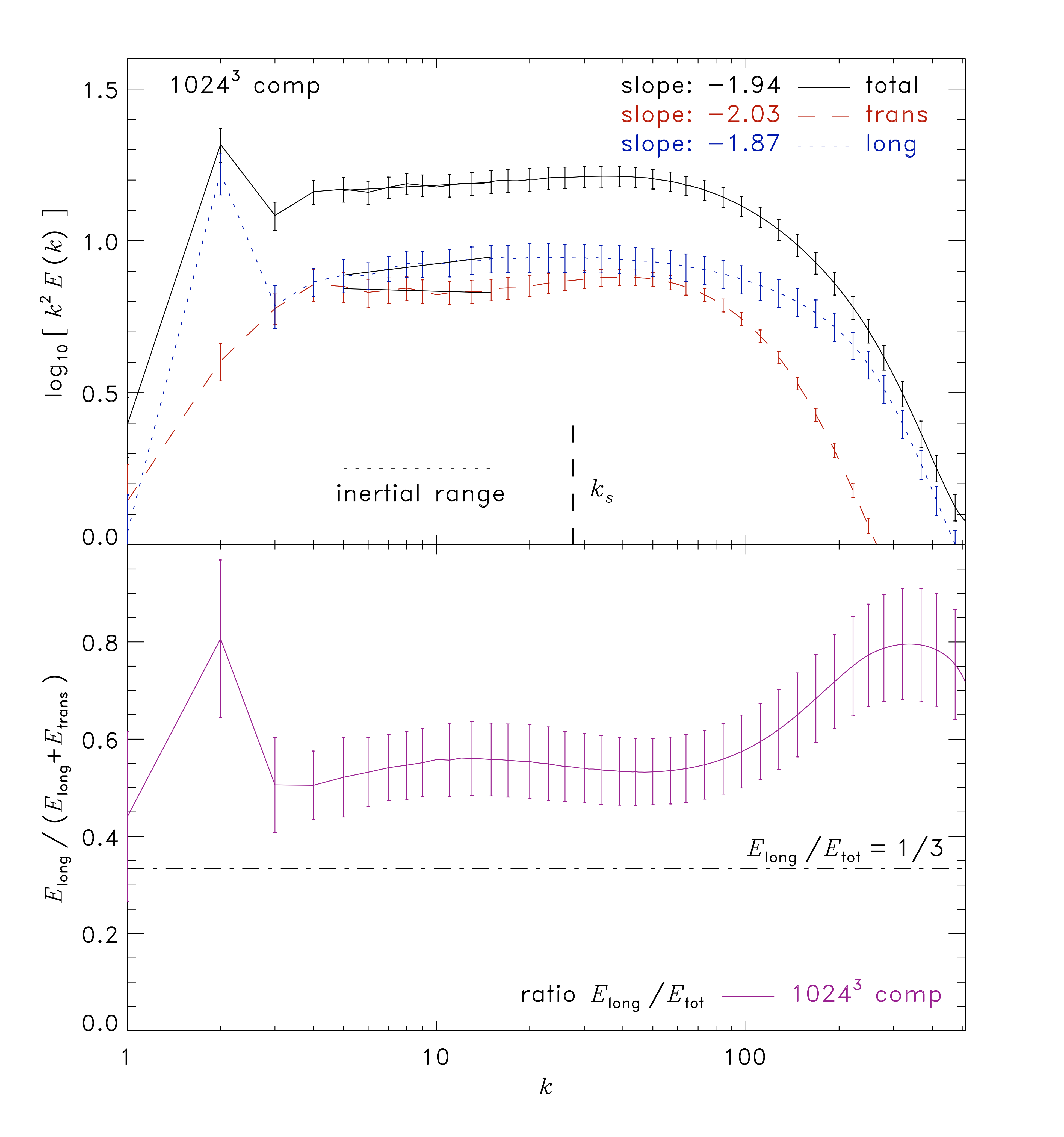
Supersonic turbulence

Velocity statistics

- Kolmogorov model does not apply to flows with $\mathscr{M} \gg 1$, because shocks form, and shocks are not local in Fourier space
- Can still estimate $P(k)$ from simple heuristic 1D argument, however
- Velocity near a shock is a step function: $v(x) = v_0$ for $x < 0$, $-v_0$ for $x > 0$
- Fourier transform is $\tilde{v}(k) = \frac{1}{\sqrt{2\pi}} \int v(x) e^{-ikx}\, dx = \sqrt{\frac{8}{\pi}} i \frac{v_0}{k} \propto 1/k$
- Power spectrum is $P(k) \sim |\tilde{v}|^2 \propto 1/k^2$
- Implied LWS is $\sigma \propto \ell^{1/2}$

Numerical evidence for k^{-2} spectrum

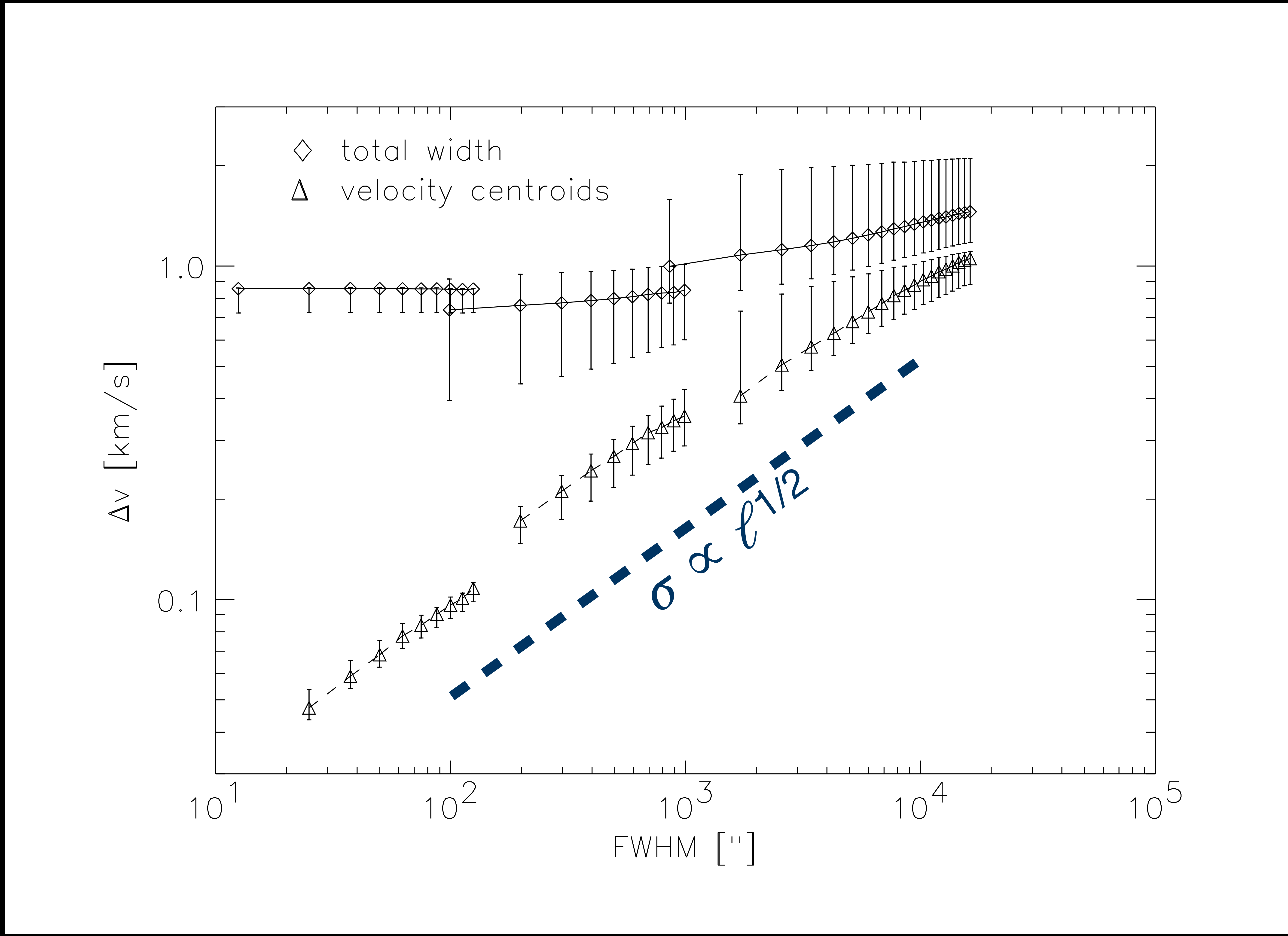
From simulations



Federrath+ 2010

Observational evidence for k^{-2} spectrum

From the observed LWS



LWS of the Polaris Flare Cloud; Ossenkopf & MacLow 2002

Supersonic density statistics

- In supersonic turbulence, shocks and rarefaction waves cause variations in density; basic question: what is the PDF of densities?
- Heuristic argument: if gas is isothermal, a shock of Mach number $\mathcal{M}$ causes density to increase by a factor of $\mathcal{M}^2$; similarly for rarefactions
- If passage of shocks and rarefactions is random, density at any given location is set by repeated multiplication by random numbers — equivalent to repeated addition of random numbers in log density
- Central limit theorem: the sum of a large number of random numbers is Gaussian-distributed, regardless (within some limits) of the shape of the distribution from which those random numbers are drawn

The lognormal density PDF

- Central limit theorem argument suggests Gaussian PDF in log of density:

$$p(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s-s_0)^2}{2\sigma_s^2}\right], \qquad s = \ln(\rho/\overline{\rho})$$

- Since $\overline{\rho} = \int p(s)\rho\, ds = \overline{\rho}\int p(s)e^s\, ds$, it is easy to show that $s_0 = -\sigma_s^2/2$
- From numerical experiments by Federrath (2013) and others, dispersion known to be related to Mach number, magnetic field strength (next class), and amount of turbulent power in compressive vs. solenoidal modes:

$$\sigma_s^2 \approx \ln\left(1 + b^2\mathcal{M}^2 \frac{\beta_0}{\beta_0+1}\right)$$

β_0 = ratio of thermal to magnetic energy density

b ~ 1/3 for purely solenoidal turbulence, ~1 for purely compressive, ~1/2 for natural mix

Mass versus volume PDFs

- PDF $p(s)$ describes distribution in volume; can also do PDF by mass
- Consider a volume V with a volume PDF $p(s)$; mass in density range s to $s+ds$ occupies a volume $dV = p(s)\, V$, and therefore has mass $dM = \rho\, p(s)\, dV$
- Substitute in:

$$\begin{aligned} dM &= \bar{\rho} e^s \cdot \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s-s_0)^2}{2\sigma_s^2}\right] dV \\ &= \bar{\rho} \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s+s_0)^2}{2\sigma_s^2}\right] dV \end{aligned}$$

- Mass PDF is therefore

$$p_M(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s+s_0)^2}{2\sigma_s^2}\right]$$

Exercise

For a region with $\mathscr{M} = 10$, no magnetic field, natural turbulent mix:

- What fraction of the volume is occupied by material at $>100 \times$ the mean density?
- What fraction of the mass is at $>100 \times$ the mean density?
- What is the mass-weighted mean density?