Class 23: Dust processes in late-stage discs

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Mark Krumholz

Outline

- Context: why think about solid particle dynamics?
- Physics of solid particles in a disc
  - Gravitational forces
  - Drag forces and drag regimes
- Dynamics in the radial direction
  - Radial drift
  - The meter barrier
- Dynamics in the vertical direction
  - Grain settling
  - KH instability
- Grain growth

Why solid particle dynamics are interesting

Why we think about them in late-stage discs, and not elsewhere

- In bulk ISM, length scales are big, so usually a good assumption that grains and gas are well-coupled when averaging over them
- Discs much smaller, but during main accretion phase, no time for separation:
  - For 0.1 $M_\odot$ accreting at ~10^{-5} $M_\odot$ / yr, mass cycles through in ~10 kyr
  - This is too short for most processes that affect grains and gas differentially
- During class II / transition disc phase, accretion rate drops to ~10^{-7} to 10^{-8} $M_\odot$ / yr; even for smaller discs, ~few × 0.01 $M_\odot$, residence time ~Myr
- Observationally, few Myr is also timescale on which discs are dispersed (see next class), so there is time for a lot more processing

The minimum mass solar nebula

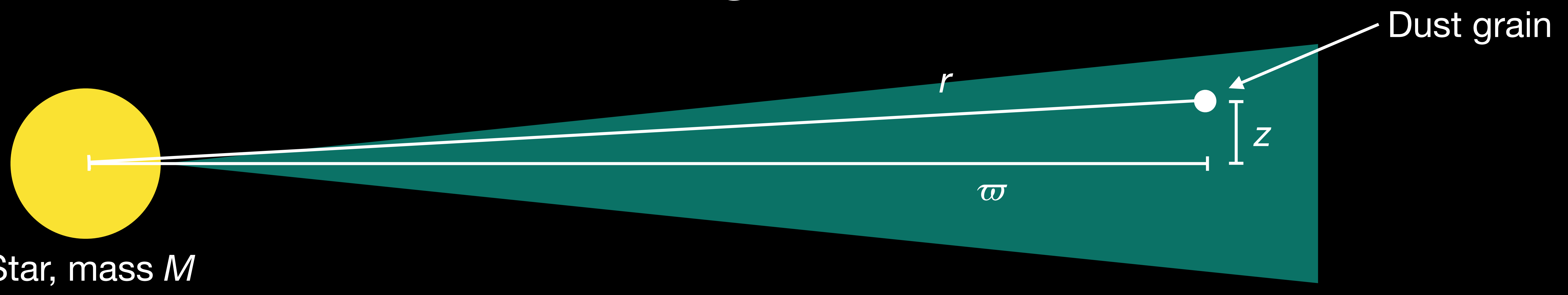
Numbers and context for analysis

- MMSN is what you get if you grind up all the planets in the Solar System, distribute them in a smooth powerlaw, and add enough H/He to match ISM abundances
- Total mass ≈ 0.01 $M_{\odot}$, but this is a minimum: Sun's disc could have been bigger, because not all heavy elements necessarily had to end up in the planets
- All material in disc: $\Sigma = \Sigma_0 \, \varpi_0^{-3/2}$, where $\varpi_0 = \varpi/1$ AU, $\Sigma_0 = 1700$ g cm^{-2}
- Solid content: ≈ 0.5% in "rocky" materials, ≈ 1.5% in "icy" materials that are solid only in places where the temperature is ≲ 150 K
- Assuming temperature set by Solar illumination, $T \approx T_0 \, \varpi_0^{-1/2}$, with $T_0 \approx 280$ K

Forces acting on dust grains

I. Gravity from central star

- Consider grain at radius ϖ from star, height $z \ll \varpi$ (where mid plane is $z = 0$)
- Work in frame orbiting at Keplerian speed; angular velocity is $\Omega = (GM / \varpi^3)^{1/2}$
- Gravitational acceleration of grain: $\mathbf{g}_* = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^2}\left(\frac{\varpi}{r}\hat{\boldsymbol{\varpi}} + \frac{z}{r}\hat{\mathbf{r}}\right)$
- Series expand in z/ϖ: $\mathbf{g}_* = -\Omega^2\left(\varpi\hat{\boldsymbol{\varpi}} + z\hat{\mathbf{z}}\right)$
- Radial component cancels with centrifugal acceleration: $\mathbf{g}_c = \varpi\Omega^2\hat{\boldsymbol{\varpi}}$



Forces acting on dust grains

II. Gravity from disc

- If grain is above disc, approximate gravitational force from disc as being due to an infinite slab of surface density Σ → $\mathbf{g}_d = -2\pi G\Sigma\hat{\mathbf{z}}$
- For a grain at $z \sim H \sim c_g^2 / \Omega$, ratio of disc and stellar acceleration is

$$\frac{g_{z,*}}{g_{z,d}} = \frac{\Omega^2 z}{2\pi G\Sigma} = \frac{c_g \Omega}{2\pi G\Sigma} = \frac{Q}{2}$$ ← Toomre Q of gas disc

- For MMSN, $Q \sim 50$ → disc gravity negligible compared to stellar gravity, can generally ignore unless disc is so massive as to be self-gravitating
- Warning: this conclusion may change when dust starts settling, because it can get much more highly concentrated; to be discussed shortly

Forces acting on dust grains

III. Aerodynamic drag — Epstein limit

- Consider grain of radius $s \ll$ particle mfp moving through gas of density ρ and sound speed c_g with speed $v \ll c_g$ relative to gas
- Number density of gas particles is $n = \rho / \mu m_\mathrm{H}$, so rate at which gas particles strike grain surface is $4\pi s^2 (\rho / \mu m_\mathrm{H}) c_g$
- Drag force exists because collisions on the front face of moving grain have higher average speed than forces on back face: mean $\Delta p = \mu m_\mathrm{H} v$;
- Net drag force: $F_D = C_D s^2 \rho v c_g, \quad C_D = \frac{4\pi}{3}$ ← $4\pi/3$ from assuming all collisions are elastic, and particles reflect in random direction — integrate over Maxwellian particle velocities and directions

Coefficient of drag

Physics exercise: Epstein drag applies for particle speed $\ll$ gas sound speed. How does the analysis change for the opposite limit, particle speed $\gg$ sound speed? Derive the approximate drag law in this limit.

Forces acting on dust grains

III. Aerodynamic drag — Stokes limit

- Epstein drag applies to grains smaller than gas mfp; in disc with particle density *n*, mfp ~ $1/n\sigma$ ~ 1 m for *n* ~ 10^{12} cm^{-3}, so this is basically all grains
- For larger bodies need to consider hydrodynamic effects: viscous forces change velocity distribution of particles near surface of body, so can no longer assume Maxwellian distribution at zero speed
- For body much larger than mfp, Stokes drag law applies: $F_D = C_D s^2 \rho v^2$
- Drag coefficient depends on viscosity of medium through which body moves

The stopping time

Basic parameter for grains

- It is informative to ask how long it will take drag to force a grain into moving at the same speed as the surrounding fluid: define the stopping time as

$$t_s = \frac{mv}{F_D} = \frac{4\pi\rho_s s^3 v}{3F_D} = \frac{s\rho_s}{c_g\rho} \text{ (Epstein)}$$

Density of grain material: ~3 g cm^{-3} for rocky composition, ~1 g cm^{-3} for icy composition

- Define dimensionless stopping time as $T_s = t_s\Omega$

- Meaning: bodies with $T_s \ll 1$ forced to match gas velocity in $\ll 1$ orbit

- For $n = 10^{12}$ cm^{-3} and $\Omega = 2\pi$ / 1 yr, $T_s \sim 1$ for $s \sim 1$ m; rule of thumb: at 1 AU, anything small enough to call a “grain” rather than a “boulder” has a short stopping time — but stopping times are much larger further out in disc!

Radial drift

Overview for small grains

- Gas in disc orbits close to Keplerian speed, but slightly slower, because some radial support is provided by gas pressure gradient
- Grains suspended in the disc do not feel pressure forces, so they need to orbit at Keplerian speed to stay at constant radius
- However, if stopping time is short, grain cannot orbit at speed different than gas speed — drag stops it
- Net effect: grains are always orbiting slightly below Keplerian speed, and thus gradually drift inward toward star

Radial drift

Deviation from Keplerian rotation

- Net radial force per unit mass on gas: $f_\varpi = -\frac{GM}{\varpi^2} - \frac{1}{\rho}\frac{\partial P}{\partial \varpi}$
- Rewrite in terms of sound speed instead of pressure:

$$f_\varpi = -\frac{GM}{\varpi^2} - \frac{P}{\rho\varpi}\frac{\partial \ln P}{\partial \ln \varpi} = -\frac{GM}{\varpi^2} - \frac{c_g^2}{\varpi}\frac{\partial \ln P}{\partial \ln \varpi} \equiv -\frac{GM}{\varpi^2} + n\frac{c_g^2}{\varpi}$$

Index of pressure profile: $P \sim r^{-n}$

- This acceleration must match gas centripetal acceleration, so gas speed is:

$$\frac{v_g^2}{\varpi} = \frac{GM}{\varpi^2} - n\frac{c_g^2}{\varpi} = \frac{v_K^2 - nc_g^2}{\varpi} \implies v_g^2 = v_K^2 - nc_g^2$$

Keplerian speed, $(GM / \varpi)^{1/2}$

- For $c_g \ll v_K$, can Taylor expand: $\Delta v = v_K - v_g \approx \frac{nc_g^2}{2v_K}$

Radial drift

Drift velocity and drift time for small bodies

- For grains with $T_s \ll 1$, grain forced by drag to orbit at same speed as gas, so grain drifts inward at speed v
- Solve for drift speed by requiring radial force balance; total force is

$$F = -\frac{GMm_s}{\varpi^2} + \frac{m_s v_g^2}{\varpi} - \frac{4\pi}{3} s^2 \rho v c_g$$

Gravity

Centrifugal force

Drag force

- Substitute in for gas rotation speed, re-arrange: $F = -\frac{c_g}{m_s}\left(\frac{n c_g}{\varpi} + \frac{v\rho}{s\rho_s}\right)$
- Solve: $v = -n c_g \frac{s\rho_s}{\varpi\rho}$ $\quad t_{\mathrm{drift}} = \frac{\varpi}{-v} = \frac{\varpi^2\rho}{n c_g s \rho_s}$ ← Time required for grain to drift into star

Radial drift as a function of body size

Extending the small body analysis

- For Epstein regime (small bodies), we have shown $v \sim s$ and $t_{\mathrm{drift}} \sim 1/s$
- For larger bodies, drag switches to Stokes drag — repeating analysis for Stokes drag law, v increases more quickly with s, but same basic effect: bigger bodies drift faster
- Regime changes again for bodies large enough that $T_s \gtrsim 1$:
    - Drag is unable to force body to orbit with gas; it orbits at Keplerian speed
    - Body still drifts inward, because faster drift than gas implies it feels a tangential drag force that causes loss of angular momentum
    - $t_{\mathrm{drift}} \sim mv / F_D \sim s^3 / s^2 \sim s$: bigger bodies have longer drift time

Implications of drift analysis

The meter barrier

- Drift time decreases with s at small s, increases at large s
- Implication: there must be an intermediate s where drift time is at a minimum
- Quantitative calculation of various drag and stopping time regimes implies minimum occurs for $s \sim 1$ m
- Drift time at this size is short: for 1 m body at mid plane of MMSN at 1 AU, $t_{\mathrm{drift}} \sim 5000$ yr
- This is known as the meter barrier: hard for bodies to grow past 1 m in size, because at this size they quickly fall into the star

Physics question: thus far we have assumed $n > 0$, i.e., gas pressure falls with radius. What happens is somewhere in the disc the opposite happens, and pressure rises with radius locally?

Grain motion in the vertical direction

Settling toward mid plane

- In vertical direction, equation of motion for grains is

$$\ddot{z} = -g_z - \frac{F_D}{m_s} = -\Omega^2 z - \frac{\dot{z}}{t_s} = \Omega\left(\Omega z - \frac{\dot{z}}{T_s}\right)$$

- This is the equation of a damped harmonic oscillator; solution is

$$z = z_0 e^{-t/\tau}, \quad \tau = 2t_s\left[1 - \left(1 - 4T_s^2\right)^{1/2}\right]^{-1}$$

- $T_s > 1/2 \rightarrow \tau$ is complex: grain performs damped oscillation about mid plane
- $T_s < 1/2 \rightarrow \tau$ is purely real, grain exponentially drifts toward mid plane
- For $T_s \ll 1$ (all objects ≪ 1 m), $\tau \approx 4\ t_s / T_s^2 \sim 100$ yr / (s / 1 cm) at 1 AU

Implications of settling

The importance of turbulence

- Settling time is ~1 Myr for ~1 μm grains, but anything bigger than that quickly sinks to mid plane, over time $\ll$ disc lifetime
- Limit on settling: if gas is turbulent, turbulent motions will exert drag forces on dust, which will pull it up out of mid plane
- MRI can cause turbulence, but late stage discs probably too neutral for it
- However: there is an unavoidable source of turbulence: if enough dust settles to make a thin dust-dominated layer, dust layer will start to rotate at Keplerian speed, faster than gas above it; this will cause KH instability

KH stability

Quantitative calculation

- For gas only, rotation speed including pressure is $v_\phi = \left(1 - \frac{nc_g^2}{2v_K^2}\right) v_K \equiv (1-\eta) v_K$
- Generalisation for mixed gas-dust fluid: $v_\phi = \left(1 - \eta \frac{\rho_g}{\rho}\right) v_K$ (ρ_g: Gas density; ρ: Total (gas + dust) density)
- Since dust settles towards midplane, ρ_g / ρ rises with z → shear
- KH stability controlled by contest between shear and gravity: shear destabilises, gravity stabilises
- Stability condition expressed in terms of Richardson number: unstable if

$$\mathrm{Ri} = \frac{(g_z/\rho)(\partial\rho/\partial z)}{(\partial v_\phi/\partial z)^2} < \mathrm{Ri}_c = 1/4$$

Evaluation of the stability condition

For solids near the mid plane

- Shear is: $\frac{\partial v_\phi}{\partial z} = -\frac{\eta}{z}\frac{\rho_g}{\rho}\left(\frac{\partial \ln \rho_g}{\partial \ln z} - \frac{\partial \ln \rho}{\partial \ln z}\right)$
- Near mid plane where solids are settling, gas density is nearly constant; density change mostly occurring in solids → can approximate

$$\frac{\partial \ln \rho_g}{\partial \ln z} \ll \frac{\partial \ln \rho}{\partial \ln z} \approx \frac{\partial \ln \rho_s}{\partial \ln z}$$

- Substitute into expression for Richardson number: $\mathrm{Ri} \approx \left(\frac{z}{\eta\varpi}\right)^2 \frac{\rho^3}{\rho_g^2 \rho_s}\left(\frac{\partial \ln \rho_s}{\partial \ln z}\right)^{-1}$
- If solids of surface density Σ_s settle into layer of scale height H_s, at order of magnitude $z \sim H_s$ and $\rho_s \sim \Sigma_s / H_s$ → $\mathrm{Ri} \sim \frac{(\rho_g H_s + \Sigma_s)^3}{(\eta\varpi\rho_g)^2\,\Sigma_s}$

Implications of KH stability analysis

Critical solid to gas ratio

- We have shown $\mathrm{Ri} \sim \frac{(\rho_g H_s + \Sigma_s)^3}{(\eta\varpi\rho_g)^2 \Sigma_s}$; recall system is unstable if Ri ≲ 1/4
- Expected outcome: if H_s gets too small, KH instability becomes active, starts to pick up dust and raise H_s until Ri is driven back down to ~1/4 → sets minimum dust scale height and maximum dust density
- However, note that, if Σ_s is too large, we will have Ri > 1/4 even for $H_s \to 0$; critical surface density where this happens is $\Sigma_s \sim 2\eta\varpi\rho_g \sim \eta\frac{\varpi}{H}\Sigma_g \sim \eta\frac{v_K}{c_g}\Sigma_g \sim \frac{c_g}{v_K}\Sigma_g$
- Interpretation: if ratio of solid to gas surface density exceeds ~c_g / v_K, KH instability becomes ineffective, solids can settle to arbitrarily thin layer
- MMSN does *not* meet this condition, but is within a factor of ~3 of it — interesting

Grain growth

Collision rates

- If dust density is ρ_d and grains have characteristic size s and density ρ_s, number density is $n = 3\rho_d/4\pi\rho_s s^3$
- For grains with velocity dispersion σ_s, collision time $t_{\rm coll} = \left(n\pi s^2\sigma_s\right)^{-1} = \dfrac{4\rho_s s}{3\rho_d\sigma_s}$
- Prior to settling, dust density $\rho_d \sim \Sigma_d/H$, velocity dispersion $\sigma_s \sim \sqrt{\dfrac{m_s}{m_{\rm H}}}\sigma_g \sim \sqrt{\dfrac{m_s}{m_{\rm H}}}H\Omega$
- Implication: collision time $t_{\rm coll} \sim \sqrt{\dfrac{m_s}{m_{\rm H}}}\dfrac{s\rho_s}{\Sigma_d\Omega} \sim \sqrt{\dfrac{m_s}{m_{\rm H}}}\dfrac{\Sigma_g}{\Sigma_d}\dfrac{T_s}{\Omega}$
- For a ~1 μm grain, this is a few yr, comparable to settling time → small grains should collide as they settle; collisions even faster once grains settle

Collisional growth

A route to planets?

- Small grains will stick very well due to van der Waals forces; probably easy to grow grains up to sub-mm sizes
- Getting to larger sizes much harder:
  - Collision rate scales as $m_s^{-1/2}\ T_s^{-1} \sim s^{-5/2}$
  - Larger grains less likely to stick, more likely to erode
  - Meter barrier makes residence time increasingly short as grains grow
- Implication: to grow bodies large enough to survive in disc for long times (~1 km), probably need some mechanism to concentrate grains: streaming instability, self-gravity due to formation of a thin layer, something like that