Class 16: Discs and outflows: theory

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Outline

- Disc formation: the importance of magnetic fields
  - The magnetic braking problem
  - Possible solutions
- Theory of accretion discs
  - Viscous disc models
  - Sources of angular momentum transport
- Theory of outflows
  - Magnetocentrifugal launching
  - Angular momentum of the wind

Magnetic braking and disc formation

General considerations

- We showed that for collapse of a core of size $\sim\varpi_c$, conservation of specific angular momentum implies circularisation at $\varpi_d \sim \beta\varpi_c$
- However, magnetic fields are capable of transporting angular momentum via magnetic torques, so individual fluid elements may not conserve *j*
- Mechanism: as inner parts of core collapse, trying to conserve *j*, angular velocity has to go up, so inner parts of core rotate faster than outer parts
- This will twist magnetic field lines connecting inner and outer parts of core; the field will resist twisting, exerting a magnetic torque whose effect is to move *j* from faster rotating inner parts to slower rotating outer parts

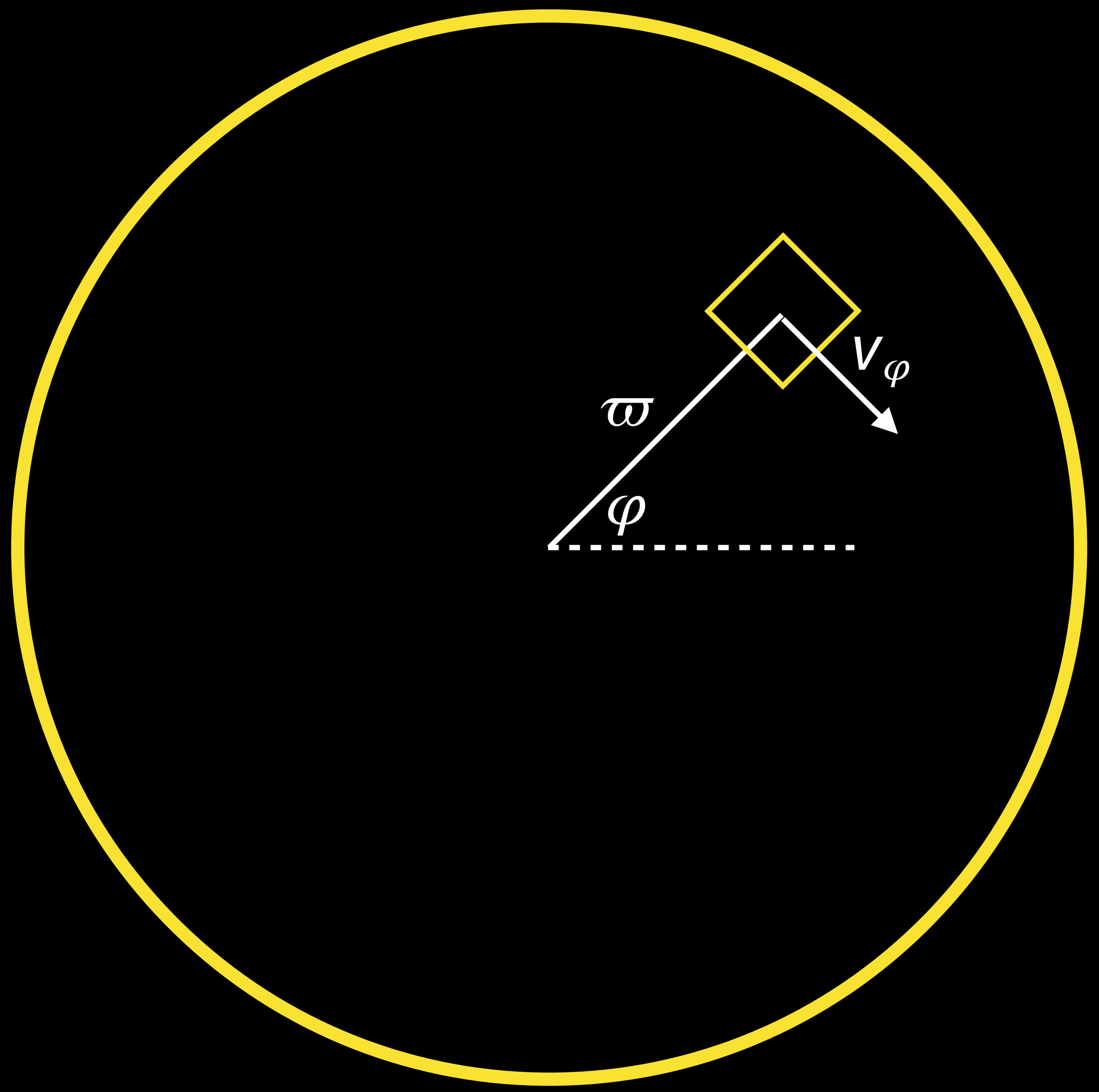
Estimate of braking effect

Part I

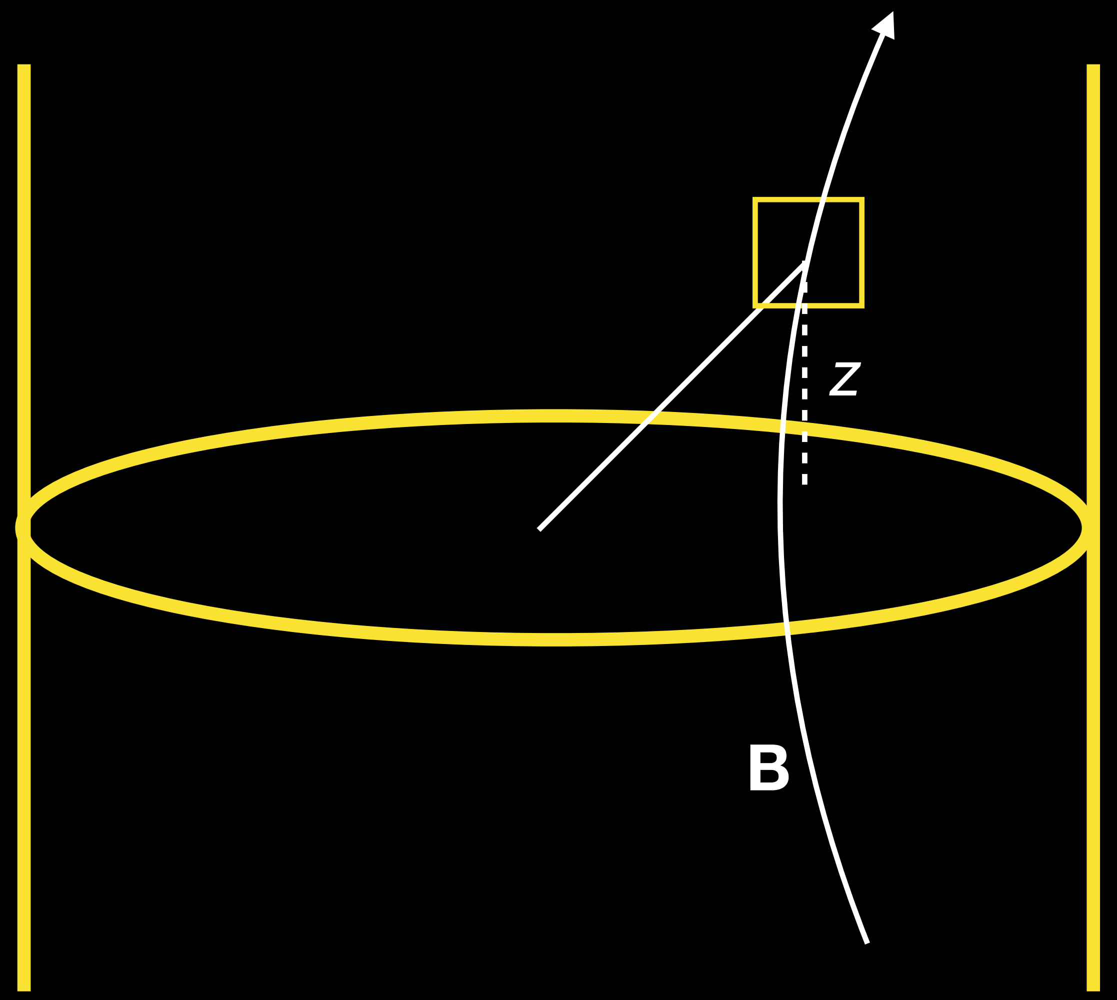
- Consider a fluid element at position (ϖ, φ, z) rotating at speed v_φ, threaded by magnetic field **B**
- For convenience separate field into poloidal part $\mathbf{B}_\mathrm{p}$ = (B_ϖ, B_z) and toroidal part B_φ
- Assuming axisymmetry, φ component of Lorentz force is

$$f_\phi = \frac{1}{4\pi}\left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right] \cdot \hat{\phi}$$
$$= \frac{1}{4\pi}\left[\frac{B_\varpi}{\varpi}\frac{\partial}{\partial \varpi}(\varpi B_\phi) + B_z \frac{\partial B_\phi}{\partial z}\right]$$
$$= \frac{1}{4\pi\varpi}\mathbf{B}_p \cdot \nabla_p (\varpi B_\phi)$$

ϖ and z parts of gradient



Top view



Side view

Estimate of braking effect

Part II

- In φ direction, $\frac{\partial}{\partial t}(\rho v_\phi) = f_\phi \quad \Longrightarrow \quad \frac{\partial}{\partial t}(\rho v_\phi \varpi) = \frac{1}{4\pi}\mathbf{B}_p \cdot \nabla_p (\varpi B_\phi)$
- Define braking time $t_{\rm br} = \frac{\rho v_\phi \varpi}{\frac{\partial}{\partial t}(\rho v_\phi \varpi)} = \frac{4\pi \rho v_\phi \varpi}{\mathbf{B}_p \cdot \nabla_p (\varpi B_\phi)}$
- Consider fluid element trying circularise at speed $v_\varphi = (GM/\varpi)^{1/2}$, where $M \sim (4\pi/3)\rho\varpi^3$ is mass already collapsed; braking time is $t_{\rm br} \sim \frac{(4\pi\rho)^{3/2} G^{1/2} \varpi^2}{\mathbf{B}_p \cdot \nabla_p (\varpi B_\phi)}$
- Assuming **B** varies on scale ϖ, so $\mathbf{B}_p \cdot \nabla_p(\varpi B_\varphi) \sim B^2$, then

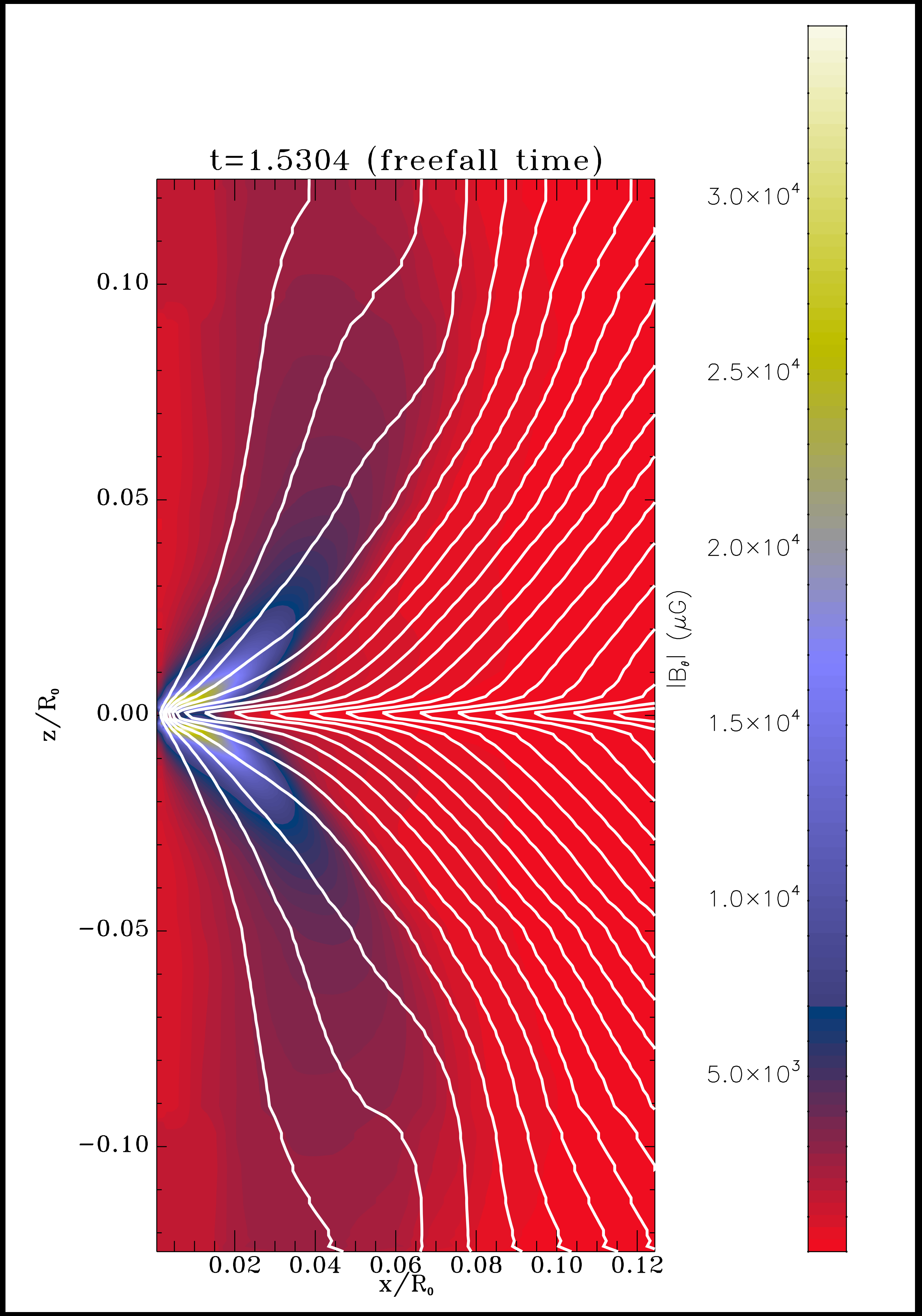
$$t_{\rm br} \sim \frac{G^{1/2}\rho^{3/2}\varpi^2}{B^2} \sim \frac{(G\rho)^{1/2}\,\varpi^2}{v_A^2} \sim \frac{t_{\rm cr}^2}{t_{\rm ff}}$$

- Bottom line: if $\mathscr{M}_A \sim 1$, so $t_{\rm cr} \sim t_{\rm ff}$, then $t_{\rm br} \sim t_{\rm ff} \rightarrow$ magnetic braking significant

The magnetic braking problem

Simulation results

- Simulations with ideal MHD show that magnetic braking is strong enough to prevent Keplerian discs from forming entirely
- Instead, pseudo-discs supported by magnetic field forms
- Problem: we observe Keplerian discs
- So how do we get around this?

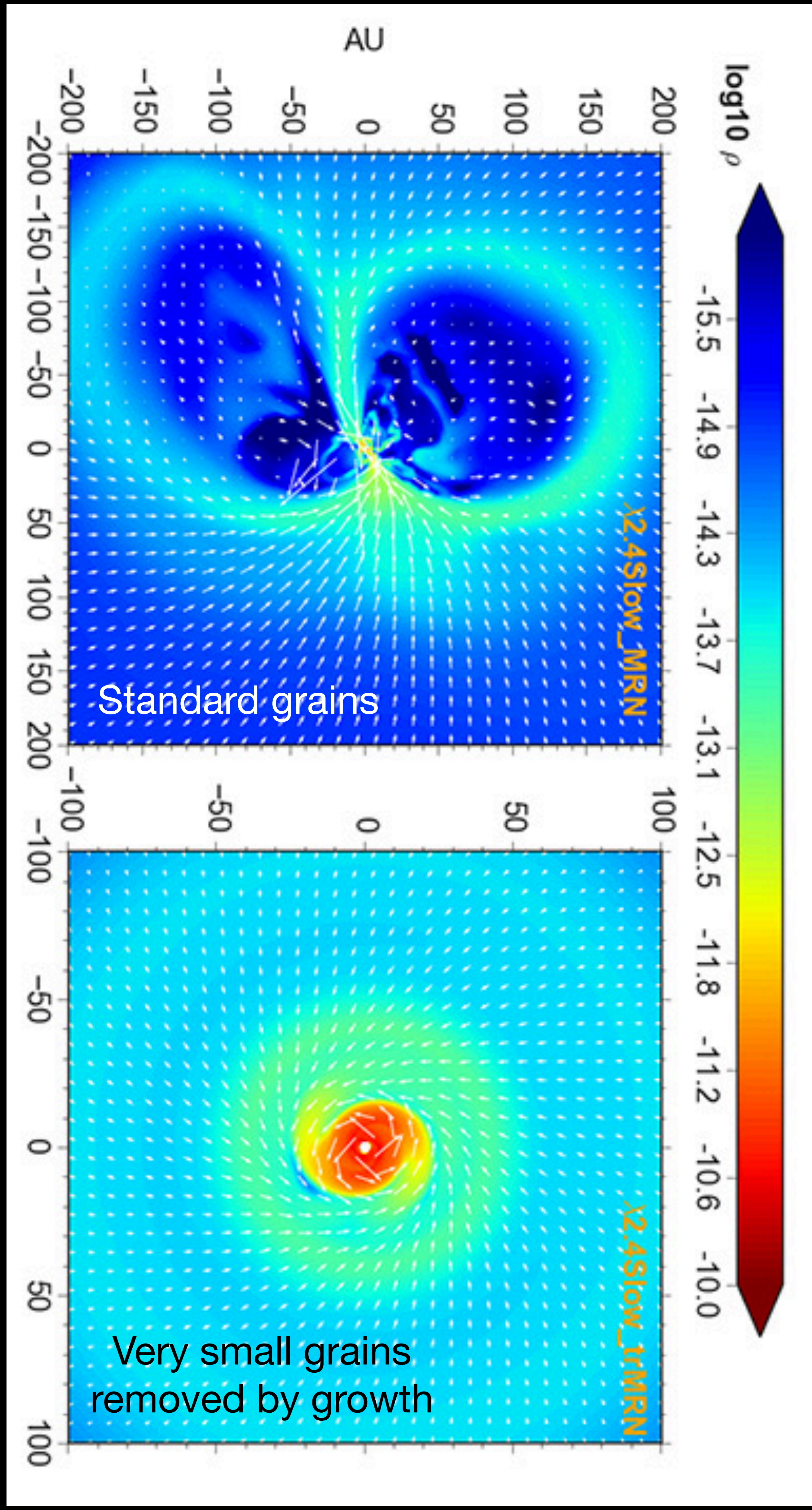


Hennebelle & Fromang 2008

Avoiding magnetic braking

Possible solutions

- Ambipolar diffusion and Hall effect may allow enough B field to escape gas to let discs form
  - Depends crucially on microphysics: at high density, charge mostly carried by dust grains
  - Conductivity depends strongly on number of very small (~1-10 nm) grains; if these are removed by growth, much less magnetic braking
- Magnetic braking much less effective when flow is turbulent, due to misalignment between magnetic torques and angular momentum



Zhao+ 2020

Evolution of discs

Mass conservation

- Consider a geometrically-thin, axisymmetric disc of surface density Σ orbiting with angular velocity Ω

- Vertically-integrated equation of mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \frac{1}{\varpi}(\varpi \rho v_\varpi) = 0 \qquad \Longrightarrow \qquad \frac{\partial \Sigma}{\partial t} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi \Sigma v_\varpi) = 0$$

- Define inward mass flux $\dot{M} = -2\pi\varpi\Sigma v_\varpi$

- Equation of mass conservation: $\dfrac{\partial \Sigma}{\partial t} - \dfrac{1}{2\pi\varpi}\dfrac{\partial \dot{M}}{\partial \varpi} = 0$

Evolution of discs

Angular momentum conservation

- Start with Navier-Stokes equation: $\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p-\rho\nabla\psi+\nabla\cdot\mathbf{T}$

  (Gravitational potential → ψ; Viscous stress tensor → $\mathbf{T}$)

- Vertically integrate: $\Sigma\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla P-\Sigma\nabla\psi+\int\nabla\cdot\mathbf{T}\,dz$

  (Vertically-integrated pressure → P)

- Write out ϕ component, using axisymmetry to drop all ϕ derivatives:

$$\Sigma\left[\frac{\partial v_\phi}{\partial t}+\frac{v_\varpi}{\varpi}\frac{\partial}{\partial\varpi}(\varpi v_\phi)\right]=\int\frac{1}{\varpi^2}\frac{\partial}{\partial\varpi}\left(\varpi^2T_{\varpi\phi}\right)\,dz$$

- Multiply by $2\pi\varpi^2$: $2\pi\varpi\Sigma\left(\frac{\partial j}{\partial t}+v_\varpi\frac{\partial j}{\partial\varpi}\right)=\int\frac{\partial}{\partial\varpi}\left(2\pi\varpi^2T_{\varpi\phi}\right)\,dz\equiv\frac{\partial\mathcal{T}}{\partial\varpi}$

  $T_{\varpi\phi}$ = Force / area
  $\varpi T_{\varpi\phi}$ = Torque / area

- Here $j=\varpi v_\phi$ is the specific angular momentum, and $\mathcal{T}=2\pi\varpi\int\varpi T_{\varpi\phi}\,dz$ is the viscous torque exerted by one ring of fluid on its neighbour

Steady viscous discs

Torque and viscosity

- If gas is in Keplerian orbit at all times, then $j = \sqrt{GM_* \varpi}$ $\quad \frac{\partial j}{\partial \varpi} = \frac{1}{2}\sqrt{\frac{GM_*}{\varpi}}$ $\quad \frac{\partial j}{\partial t} = 0$
- Plug into equations: $\dot{M} = -\left(\frac{\partial j}{\partial \varpi}\right)^{-1} \frac{\partial \mathcal{T}}{\partial \varpi}$ → viscous torque sets accretion rate
- Stress $T_{\varpi\phi}$ scales with rate of strain, defined as inverse of timescale required for adjacent fluid elements shear apart by a distance of order their separation
- For gas parcels at radii separated by dϖ, relative velocity dv_ϕ = ϖ dΩ, so rate of strain = dv_ϕ /dϖ = ϖ (dΩ / dϖ)
- Thus can write $T_{\varpi\phi} = \rho\nu\varpi \frac{d\Omega}{d\varpi} \quad \Longrightarrow \quad \mathcal{T} = 2\pi\varpi^3 \Sigma\nu \frac{d\Omega}{d\varpi}$

Viscosity

Steady viscous discs

Evolution equation and solutions

- Plugging in torque gives $$\frac{\partial \Sigma}{\partial t} = \frac{3}{\varpi}\frac{\partial}{\partial \varpi}\left[\varpi^{1/2}\frac{\partial}{\partial \varpi}\left(\nu\Sigma\varpi^{1/2}\right)\right] \qquad v_\varpi = -\frac{3}{\Sigma\varpi^{1/2}}\frac{\partial}{\partial \varpi}\left(\nu\Sigma\varpi^{1/2}\right)$$
- Can solve numerically, but illuminating to consider trivial case Σ, υ constant: $$v_\varpi = -\frac{3}{2}\frac{\nu}{\varpi} \qquad \dot{M} = 3\pi\Sigma\nu$$
- Thus υ sets accretion rate — but what is υ? Since torque has units of pressure, it is common to non-dimensionalise by scaling the pressure:

Shakura-Sunyaev α parameter → $$T_{\varpi\phi} = -\alpha p \frac{\Omega/\varpi}{d\Omega/d\varpi} \iff \nu = \alpha\frac{c_s^2}{\Omega} \iff \dot{M} = 3\pi\alpha\Sigma\frac{c_s^2}{\Omega}$$

- Note: accretion timescale $$t_{\mathrm{acc}} \sim \frac{M_{\mathrm{disc}}}{\dot{M}} \sim \alpha\frac{M}{\Sigma}\frac{\Omega}{c_s^2} \sim \alpha\frac{\Omega}{(c_s/R_{\mathrm{disc}})^2} \sim \alpha\left(\frac{t_{\mathrm{orb}}}{t_{\mathrm{cross}}}\right)^2 t_{\mathrm{orb}}$$
- Thus value of α sets ratio of accretion time to orbital time

Angular momentum transport mechanisms

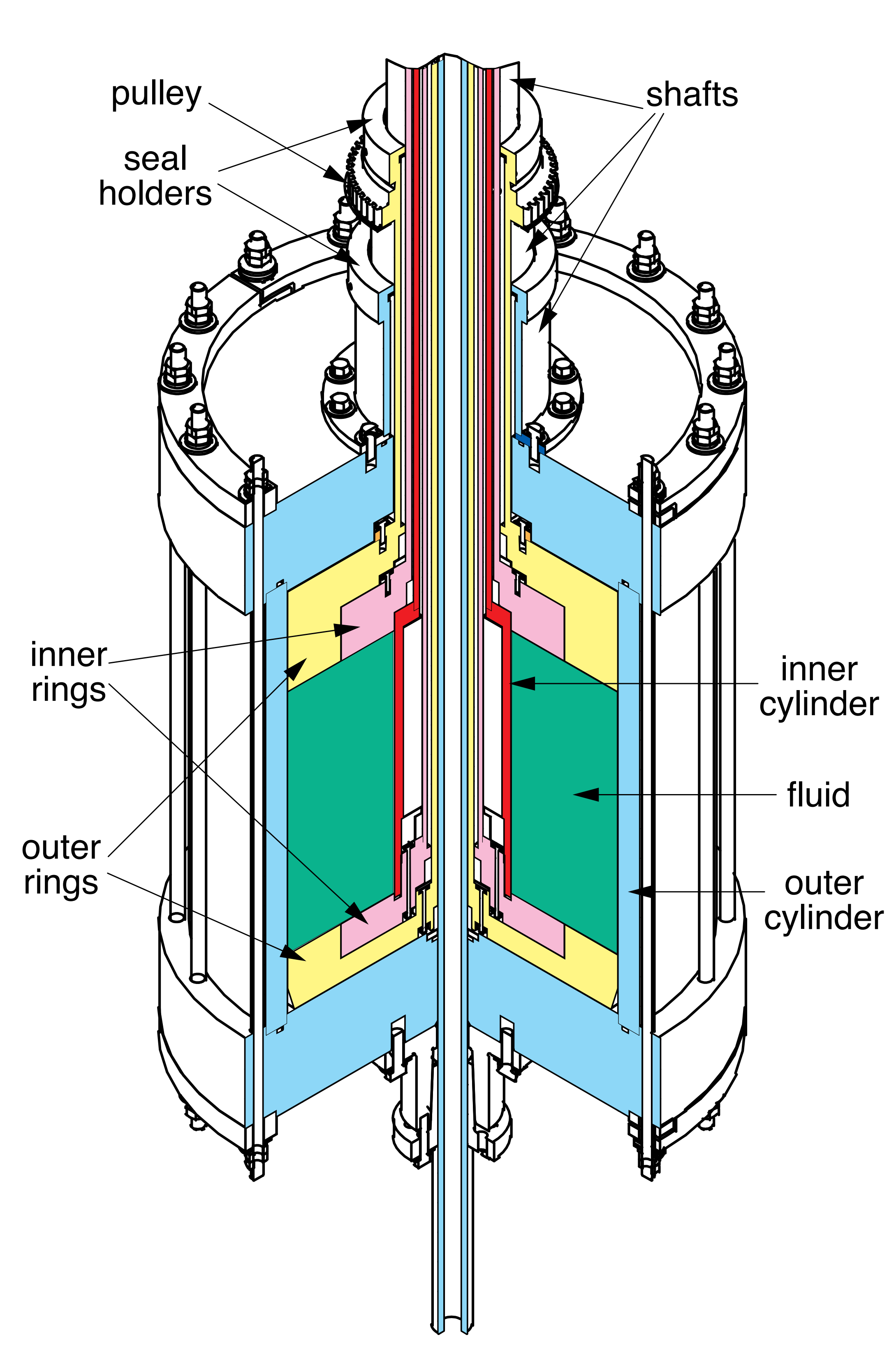
What is responsible for setting α?

- Empirically measured accretion rates on young stars suggest α ~ 0.01, but mechanism by which this is established still debated
- Cannot be ordinary fluid viscosity: $\alpha = \frac{\Omega}{c_s^2}\nu \sim \frac{\Omega}{c_s^2}\left(\frac{c_s}{n\sigma}\right)$ (c_s: Sound speed; $n\sigma$: Number density × cross section)
- Plugging in Ω ~ 1/yr, c_s ~ 1 km/s, n ~ 10^{12} cm^{-3}, σ ~ (1 nm)^2 gives α ~ 10^{-10}. Accretion time would be longer than Hubble time!
- Other possibilities: (1) hydrodynamic turbulence, (2) MHD turbulence, (3) gravitational torques, (4) winds

Hydrodynamic turbulence

Doesn’t work

- Viscosity is a diffusion-like term: measures mean distance something (momentum or angular momentum) travels, multiplied by speed with which it travels
- Molecular viscosity negligible because particle mean free path is small, but turbulence can create coherent motions on scales up to ~disc scale height — could in principle transport angular momentum faster
- However, both simulations and laboratory experiments show little *j* transport by hydrodynamic turbulence: $\alpha \lesssim 10^{-6}$



Ji+ 2006

MHD turbulence

Magnetorotational instability (MRI)

- Basic mechanism: fluid elements tethered by magnetic field line, one closer to star
- Element closer to star tries to rotate faster, move ahead of slower, more distant one
- Field resists being sheared apart, pulls back on inner element, forward on outer element — transfers *j*
- This makes inner element slow down, so it falls even further inward; outer element goes outward: force gets even bigger → instability

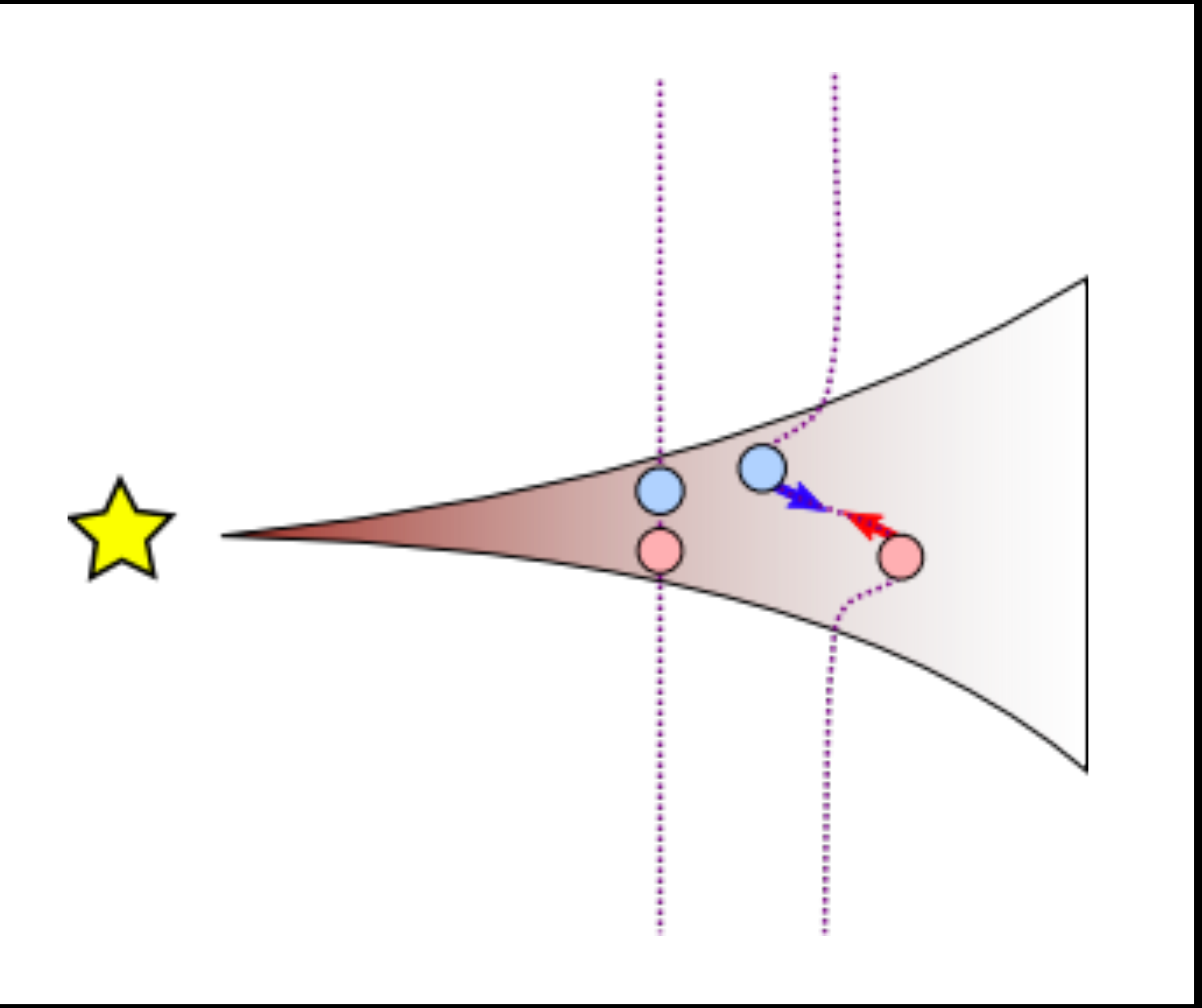


Diagram from Harvard Ay201b "book"

MRI advantages and disadvantages

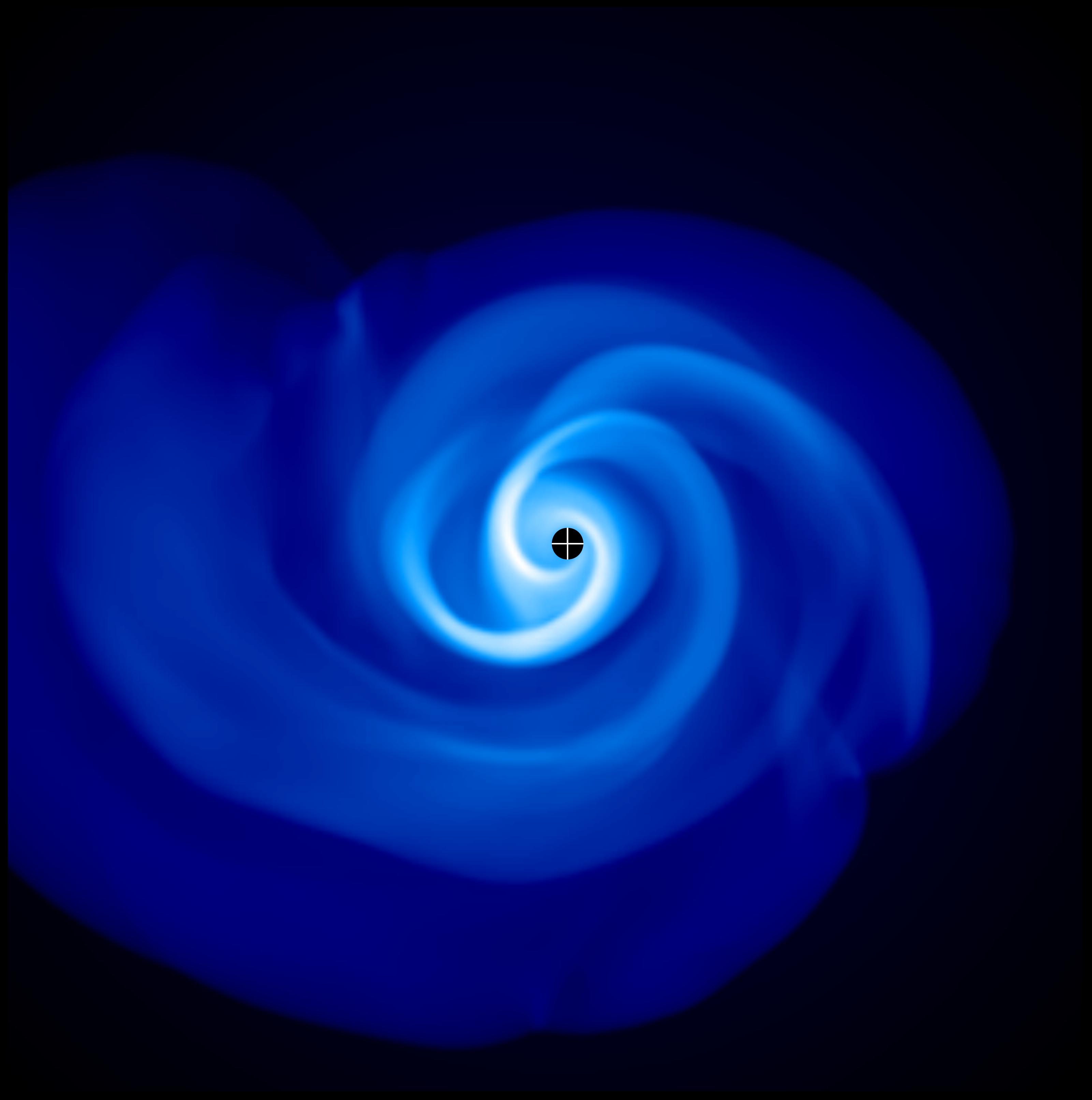
Still undecided if this is the right model

- Simulations show MRI provides about the right value of α
- Big uncertainty is coupling of field to gas: high density in discs, so ion fraction likely to be very, very low — B field may not be coupled enough to gas to allow MRI to take place
- May also depend on height within disc: surface layers exposed to stellar radiation could be ionised and MRI-active, while mid plane has lower ionisation and be an MRI “dead zone”

Gravity-driven accretion

Discs near Q = 1

- For discs massive enough to be self-gravitating, instabilities produce spiral arms
- These efficiently transport angular momentum (more in paper by Kratter+)
- May be dominant accretion process during main accretion phase of star, when accretion rate is high and disc is massive
- However, accretion also observed in class II systems where disc is at $Q \gg 1$; that can't be GI

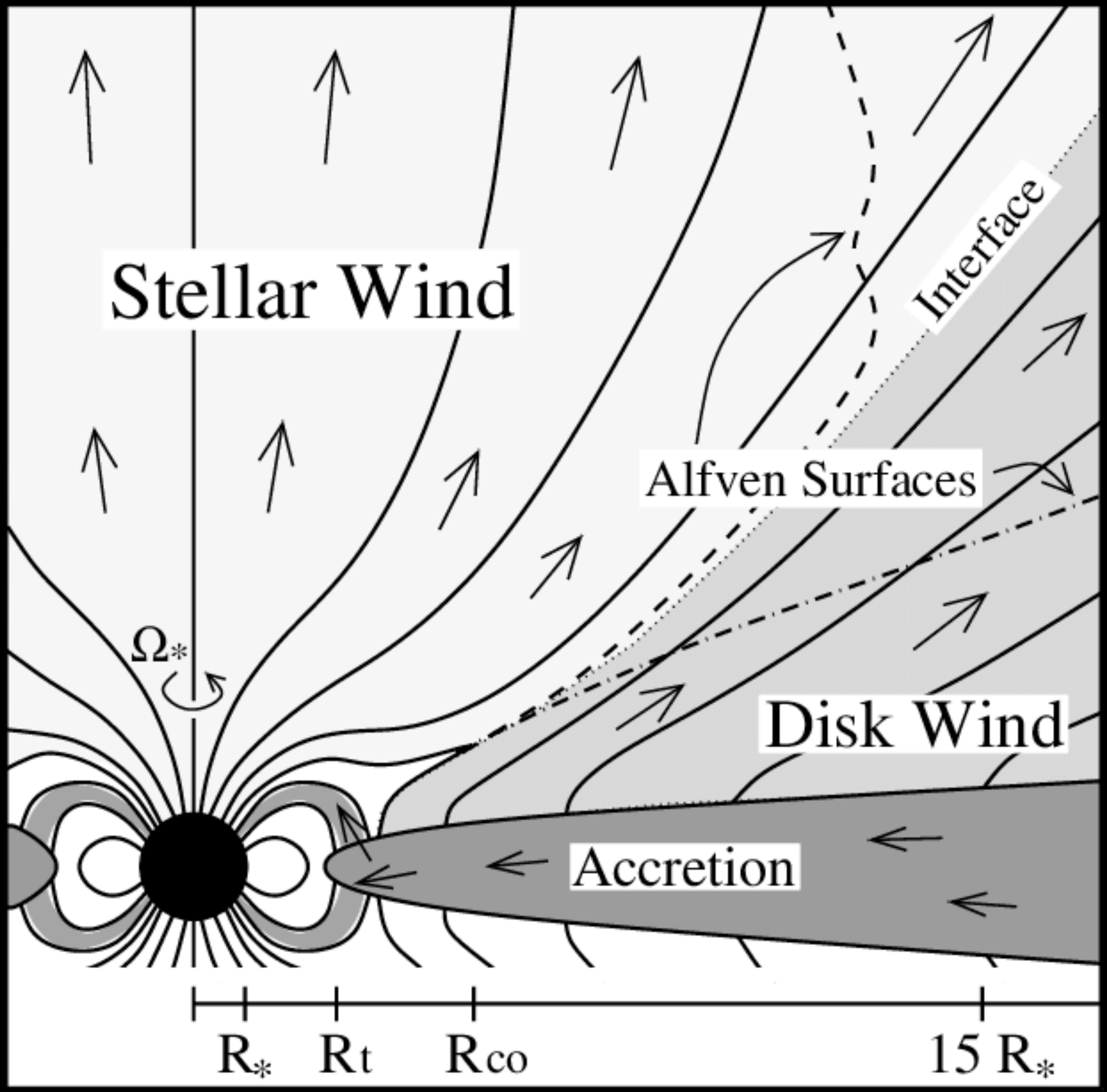


Kratter+ 2010

Discs winds and wind-driven accretion

Basic considerations

- Magnetised accretion discs generically drive winds
- Power source distinct from stellar winds, which are given by thermal or radiation pressure, and ultimately power by star
- Disc winds are powered by energy released by accretion
- This potentially makes them much more powerful than ordinary stellar winds



Matt & Pudritz (2005)

Theory of disc winds

The basics

- Write magnetic field as sum of toroidal and poloidal parts: $\mathbf{B} = \mathbf{B}_p + B_\phi \hat{\phi}$
- Consider region slightly above disc plane; in this region magnetic pressure » gas pressure, so approximate field in this region as rigidly rotating at speed Ω
- Gas subject to two potentials (in rotating frame): $\psi = \psi_g + \psi_c = -\frac{GM_*}{\sqrt{\varpi^2 + z^2}} - \frac{1}{2}\Omega^2\varpi^2$ (Gravitational potential: ψ_g; Centrifugal potential: ψ_c)
- Assume gas is fixed to field line that is anchored to disc in mid plane at distance ϖ_0, so $\Omega = (GM_* / \varpi_0^3)^{1/2}$; then potential is

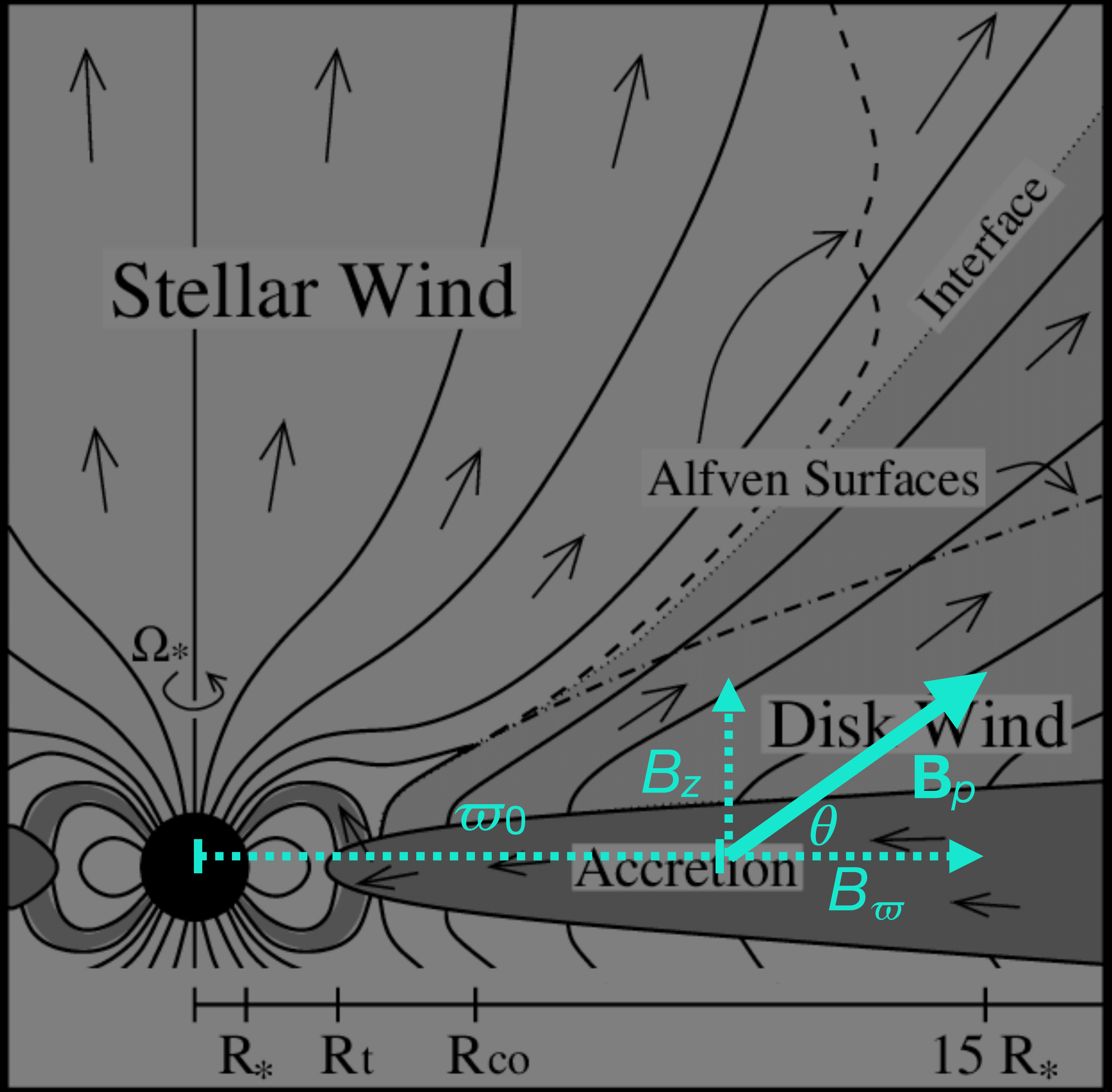
$$\psi = -\frac{GM_*}{\varpi_0}\left[\frac{1}{2}\left(\frac{\varpi}{\varpi_0}\right)^2 + \frac{\varpi_0}{\sqrt{\varpi^2 + z^2}}\right]$$

Force and stability analysis

- Force due to potential: $\mathbf{f} = -\nabla\psi = -GM_* \left\{ \varpi \left[\frac{1}{(\varpi^2 + z^2)^{3/2}} - \frac{1}{\varpi_0^3} \right] \hat{\varpi} + \frac{z}{(\varpi^2 + z^2)^{3/2}} \hat{z} \right\}$
- Define $\cos\theta = B_\varpi / \sqrt{B_\varpi^2 + B_z^2}$
- Consider fluid parcel at mid plane displaced along field by ds, to $(\varpi, z) = (\varpi_0 + \cos\theta\, ds, \sin\theta\, ds)$
- It feels force $df = \frac{GM_*}{\varpi_0^3} (3\cos\theta\hat{\varpi} - \sin\theta\hat{z})\, ds$
- Component parallel to field is:

$$df_\parallel = \frac{GM_*}{\varpi_0^3} \left(3\cos^2\theta - \sin^2\theta\right) ds$$

- Implication: if $\theta < 60°$, $df_\parallel > 0$: parcel flows out as part of a wind



Wind angular momentum and accretion

- Outflowing material held in solid-body rotation by field, so $v = \varpi\Omega$
- Clearly this must break down for sufficiently large ϖ — otherwise we would eventually have $v > c$
- Breakdown occurs when wind velocity ~ Alfvén speed, because at this point field is not “rigid” enough to keep forcing material to rotate as solid body — radius at which this occurs is called Alfvén radius ϖ_A
- Specific angular momentum of wind material is therefore $j = \varpi_A v = \varpi_A{}^2\Omega$, larger than that of disc material by $(\varpi_A/\varpi_0)^2$
- Thus wind can remove all j and allow accretion if $\dot{M}_w \geq \left(\frac{\varpi_0}{\varpi_A}\right)^2 \dot{M}_*$