Class 11: Galactic-scale star formation rates: theory

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Outline

- General considerations
  - Toomre stability
  - Vertical force balance
- The top-down approach
  - Hydrodynamics plus gravity alone
  - Feedback-regulated models
  - Challenges for the top-down approach
- The bottom-up approach
  - Thermal feedback and self-gravitating gas
  - Star formation rate in self-gravitating gas
  - Challenges for the bottom-up approach

Toomre stability

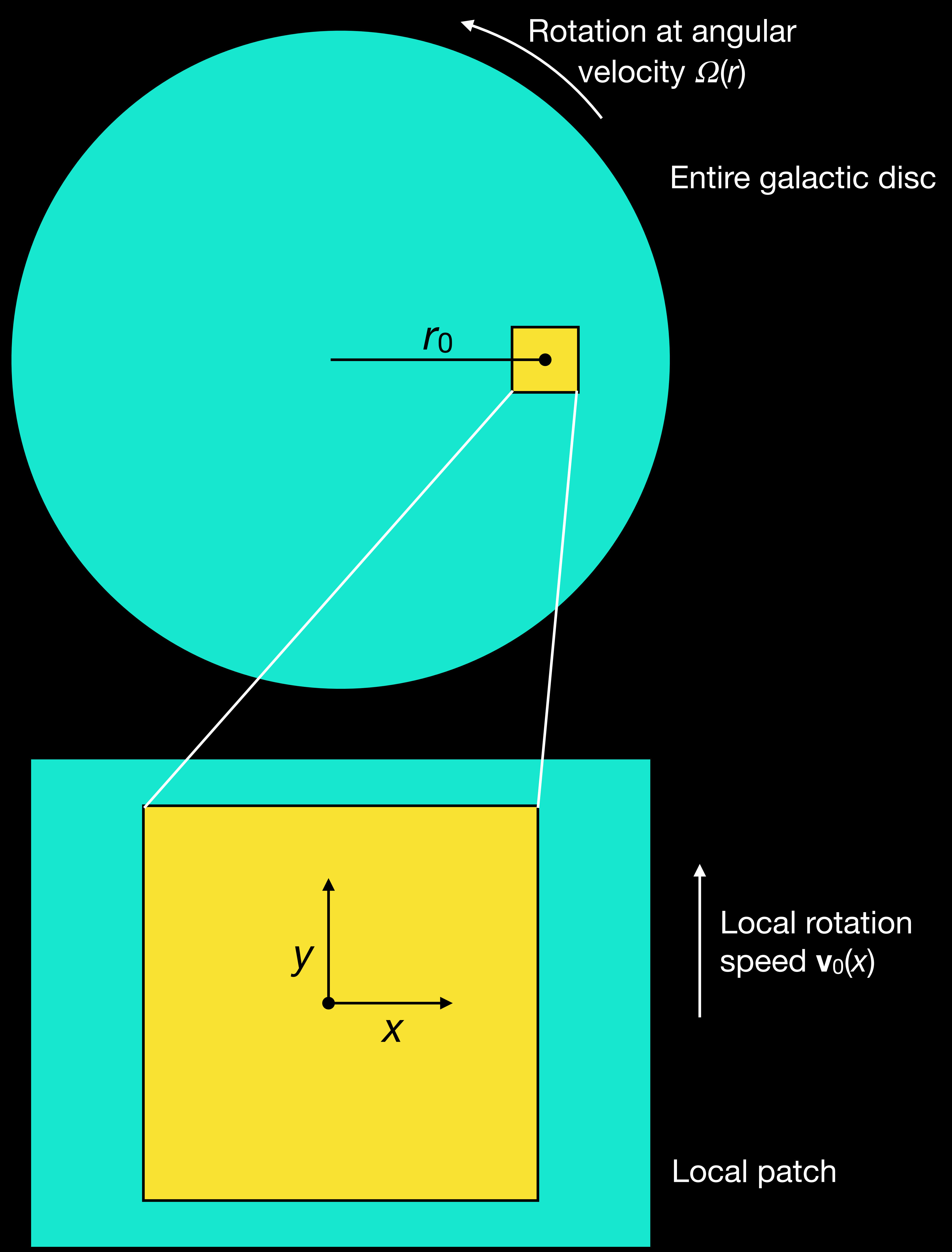
Background

- An important consideration for galactic discs: under what conditions is a galactic disc stable against self-gravity?
- Original calculation due to Alar Toomre (1964) for case of a thin stellar disc
- Basic setup: rotating, axisymmetric, thin disc
  - Most general versions of the calculation include stars, finite gas cooling time, finite thickness, non-axisymmetric modes
  - We will do the zeroth-order version: gas only (no stars), isothermal (no cooling), infinitely thin, axisymmetric modes only
  - Lots of papers extending to more general cases

Toomre stability

Background state

- Consider an infinitely thin, axisymmetric gas disc of surface density $\Sigma(r)$ occupying the plane $z = 0$; it rotates with angular velocity $\boldsymbol{\Omega}(r) = \Omega\,\hat{z}$; gas is isothermal with sound speed c_s, and starts at rest (apart from rotation)
- Work in reference frame co-rotating with disc at radius r_0; set up Cartesian coordinate system so x = radially outward, y = direction aligned with rotation, origin = centre of local box



Toomre stability

Basic equations

- In the coordinate system we have just describe, equations of motion are:

Mass conservation

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{v}) = 0$$

Momentum conservation (Pressure, Gravity, Coriolis, Centrifugal)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = \underbrace{-c_s^2 \frac{\nabla \Sigma}{\Sigma}}_{\text{Pressure}} \underbrace{- \nabla \phi}_{\text{Gravity}} \underbrace{- 2\boldsymbol{\Omega} \times \mathbf{v}}_{\text{Coriolis}} + \underbrace{\Omega^2 \left(x\hat{x} + y\hat{y} \right)}_{\text{Centrifugal}}$$

Poisson equation

$$\nabla^2 \phi = 4\pi G \Sigma \delta(z)$$

δ function since disc is infinitely thin

Toomre stability

Linearised equations

- Background state plus perturbation: $\Sigma = \Sigma_0 + \varepsilon\Sigma_1$, $\phi = \phi_0 + \varepsilon\phi_1$, $\mathbf{v} = \mathbf{v}_0 + \varepsilon\mathbf{v}_1$
- Substitute in and linearise system, as for Jeans instability:

$$\frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \nabla \cdot \mathbf{v_1} + \nabla \cdot (\Sigma_1 \mathbf{v}_0) = 0$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_0 = -c_s^2 \frac{\nabla \Sigma_1}{\Sigma_0} - \nabla \phi_1 - 2\boldsymbol{\Omega} \times \mathbf{v}_1$$

$$\nabla^2 \phi_1 = 4\pi G \Sigma_1 \delta(z)$$

- Centrifugal term cancels because gravity = centrifugal force in rotating frame
- Cannot set $\mathbf{v}_0 = 0$, because this holds at $x = y = 0$, but not elsewhere; need to keep because derivatives of $\mathbf{v}_0$ are non-zero

Toomre stability

Fourier analysis I

- Try axisymmetric Fourier mode, modified in z direction: $\phi_1 = \phi_a\, e^{i(kx - \omega t) - |kz|}$
- Substitute into Poisson eqn: $\phi_a \nabla^2 e^{i(kx-\omega t)-|kz|} = 4\pi G \Sigma_a e^{i(kx-\omega t)} \delta(z)$
- Solve by integrating both sides in z from $-\zeta$ to ζ, taking limit as $\zeta \to 0$:

$$\phi_a \lim_{\zeta\to 0} \int_{-\zeta}^{\zeta} \frac{\partial^2}{\partial z^2} e^{-|kz|}\, dz = 4\pi G \Sigma_a$$

$$\phi_a \lim_{\zeta\to 0} \left[\left(\frac{d}{dz} e^{-|kz|} \right)_{z=\zeta} - \left(\frac{d}{dz} e^{-|kz|} \right)_{z=-\zeta} \right] = 4\pi G \Sigma_a$$

$$-2\phi_a |k| = 4\pi G \Sigma_a$$

$$\phi_a = -\frac{2\pi G \Sigma_a}{|k|}$$

Toomre stability

Fourier analysis II

- Write out Ω_0 and $\mathbf{v}_0$ near target radius r_0 using first-order Taylor expansion: $\Omega \approx \Omega_0 + x\,(\mathrm{d}\Omega/\mathrm{d}r)_0$, and inertial frame $v_\phi = r\Omega \approx r_0\,[\Omega_0 + x\,(\mathrm{d}\Omega/\mathrm{d}r)_0]$, so in co-rotating frame $\mathbf{v}_0 \approx x\, r_0\,(\mathrm{d}\Omega/\mathrm{d}r)_0\, \hat{y}$

- Can now put Fourier modes into equations of mass and momentum conservation: $\Sigma_1 = \Sigma_a\, e^{i(kx - \omega t)}$, $v_{1,x} = v_{a,x}\, e^{i(kx - \omega t)}$, $v_{1,y} = v_{a,y}\, e^{i(kx - \omega t)}$

Exercise: obtain the linearised equations of mass and momentum conservation

Toomre stability

Fourier analysis II

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- Can now put Fourier modes into equations of mass and momentum conservation: $\Sigma_1 = \Sigma_a\, e^{i(kx - \omega t)}$, $v_{1,x} = v_{a,x}\, e^{i(kx - \omega t)}$, $v_{1,y} = v_{a,y}\, e^{i(kx - \omega t)}$
- Result:

Mass
$$-i\omega\Sigma_a = -ik\Sigma_0 v_{a,x}$$

x momentum
$$-i\omega v_{a,x} = -ikc_s^2\frac{\Sigma_a}{\Sigma_0} + ik\frac{2\pi G\Sigma_a}{|k|} + 2\Omega_0 v_{a,y}$$

y momentum
$$-i\omega v_{a,y} = -\left[2\Omega_0 + r_0\left(d\Omega/dr\right)_0\right] v_{a,x}$$

Toomre stability

Dispersion relation I

- This three linear equations in three unknowns; easiest to solve by rewriting in matrix form:

$$\begin{bmatrix} ik(2\pi G/|k| - c_s^2/\Sigma_0) & i\omega & 2\Omega_0 \\ 0 & -2\Omega_0 - r_0(d\Omega/dr)_0 & i\omega \\ -i\omega & ik\Sigma_0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_0 \\ v_{a,x} \\ v_{a,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Equation has a non-trivial solution only if determinant of matrix is 0 →

$$\omega^2 = \left[2\left(2 + \frac{r}{\Omega}\frac{d\Omega}{dr}\right)\Omega^2\right]_0 - 2\pi G\Sigma_0|k| + k^2 c_s^2$$

- RHS is a quadratic in $|k|$, has minimum at $|k| = \pi G \Sigma_0 / c_s^2$

Toomre stability

Dispersion relation II

- Instability exists if $\omega^2 < 0$ for any k, so plug in k that gives smallest ω^2
- Result is $\omega^2 = \left[2\left(2 + \frac{r}{\Omega}\frac{d\Omega}{dr}\right)\Omega^2\right]_0 - \left(\frac{\pi G \Sigma_0}{c_s}\right)^2 = \kappa^2 - \left(\frac{\pi G \Sigma_0}{c_s}\right)^2$

  κ = epicyclic frequency: frequency with which a star perturbed off a circular orbit oscillates around that orbit
- Condition $\omega^2 < 0$ therefore reduces to

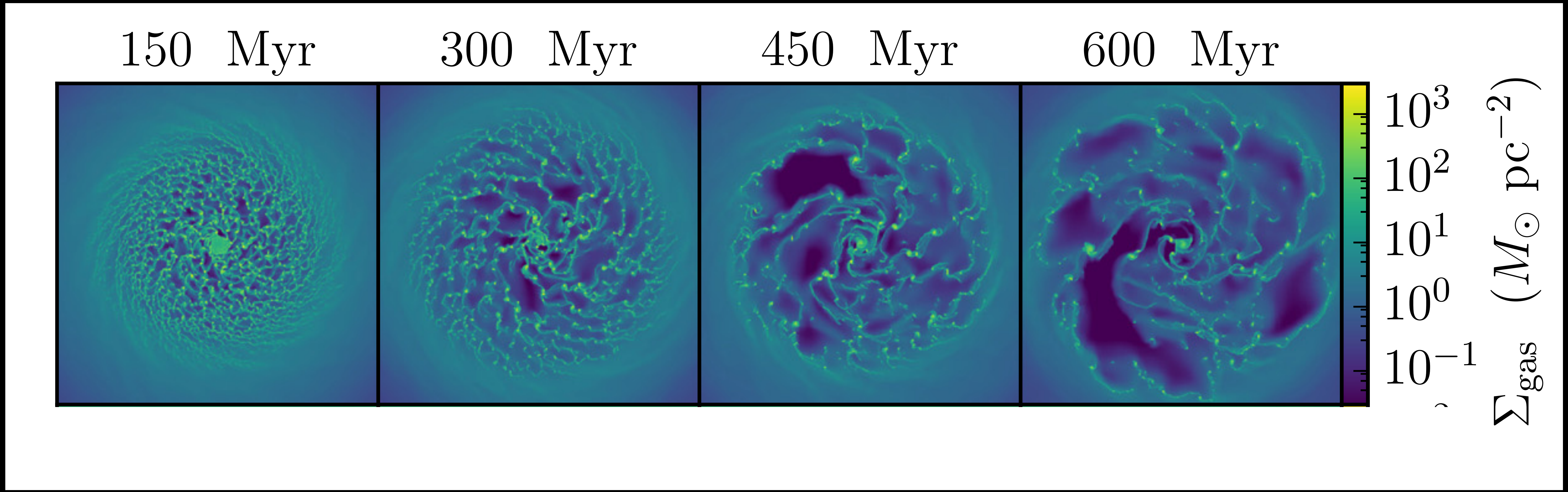
$$\frac{\kappa c_s}{\pi G \Sigma_0} \equiv Q < 1$$

- Interpretation: gas is stabilised by shear (κ term) and pressure (c_s) term, destabilised by gravity ($G\Sigma_0$ term); if gravity wins, system is unstable
- Secondary interpretation: $Q \sim 1$ ↔ free-fall time $t_{\mathrm{ff}} \sim$ orbital time Ω^{-1}

Toomre instability

Implications and questions

- If $Q < 1$, disc is unstable to develop self-gravitating rings (which then tend to break up into self-gravitating clumps)
- Observed galactic discs sit near $Q = 1$; no clear correlation of Q with star formation, but may be due to observational error given small observed range
- Mechanism for forcing $Q = 1$ not entirely clear



Simulation of a Toomre-unstable disc; Goldbaum+ 2015

Vertical force balance

General considerations

- Toomre instability describes effect of self-gravity in plane of disc
- Also important to consider force balance in the vertical direction
- Neutral gas in Milky Way disc has scale height ~150 pc, similar for other disc galaxies — this sets density at midplane, and thus is plausibly related to SFR
- Scale height likely set by balance between gravity and pressure, with significant pressure contributions from thermal pressure, turbulent pressure, magnetic pressure, and possibly cosmic ray pressure

Vertical force balance

Derivation I

- Start with momentum equation in tensor form:

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = -\nabla\cdot(\rho\mathbf{v}\mathbf{v}+P\mathbf{I}) + \frac{1}{4\pi}\nabla\cdot\left(\mathbf{B}\mathbf{B}-\frac{B^2}{2}\mathbf{I}\right) + \rho\mathbf{g}$$

Pressure plus advection Lorentz force Gravity

- Set up coordinate system with disc centred on $z = 0$ plane, take z component:

$$\frac{\partial}{\partial t}(\rho v_z) = -\nabla\cdot(\rho\mathbf{v}v_z) - \frac{dP}{dz} + \frac{1}{4\pi}\nabla\cdot(\mathbf{B}B_z) - \frac{1}{8\pi}\frac{d}{dz}B^2 + \rho g_z$$

- Consider an area A bounded by a curve S, lying at constant z; for any quantity Q, define $\langle Q\rangle = (1/A)\int_A Q\, dA$; integrate both sides over A to get

$$\frac{\partial}{\partial t}\langle\rho v_z\rangle = -\frac{1}{A}\int_A \nabla\cdot(\rho\mathbf{v}v_z)\, dA - \frac{d\langle P\rangle}{dz} + \frac{1}{4\pi A}\int_A \nabla\cdot(\mathbf{B}B_z)\, dA - \frac{1}{8\pi}\frac{d}{dz}\langle B^2\rangle + \langle\rho g_z\rangle$$

Vertical force balance

Derivation II

- Separate *xy* from *z* components in divergences:

$$\frac{\partial}{\partial t}\langle \rho v_z \rangle = -\frac{1}{A}\int_A \nabla_{xy}\cdot(\rho \mathbf{v} v_z)\, dA - \frac{d}{dz}\langle v_z^2 \rangle - \frac{d\langle P \rangle}{dz}$$
$$+\frac{1}{4\pi A}\int_A \nabla_{xy}\cdot(\mathbf{B}B_z)\, dA + \frac{1}{4\pi}\frac{d}{dz}\langle B_z^2 \rangle - \frac{1}{8\pi}\frac{d}{dz}\langle B^2 \rangle + \langle \rho g_z \rangle$$

- Apply divergence theorem to *xy* integrals:

$$\frac{\partial}{\partial t}\langle \rho v_z \rangle = -\frac{d}{dz}\langle v_z^2 \rangle - \frac{d\langle P \rangle}{dz} + \frac{1}{4\pi}\frac{d}{dz}\langle B_z^2 \rangle - \frac{1}{8\pi}\frac{d}{dz}\langle B^2 \rangle + \langle \rho g_z \rangle$$
$$-\frac{1}{A}\int_S v_z \rho \mathbf{v}\cdot\hat{\mathbf{n}}\, d\ell + \frac{1}{4\pi A}\int_S B_z \mathbf{B}\cdot\hat{\mathbf{n}}\, d\ell$$

Unit vector orthogonal to *S*

Exercise: come up with an argument why, in steady state, the line integral terms in the momentum equation should have an average value of zero.

Vertical force balance

Derivation III

- Line integral terms represent transport of z momentum across S by gas flows and by magnetic forces — but if we pick a large enough portion of a galaxy disc, these must vanish on average

- Thus in steady state, equation of vertical momentum balance reads:

$$\frac{d}{dz}\left\langle P + \rho v_z^2 + \frac{B^2}{8\pi}\right\rangle - \frac{d}{dz}\left\langle \frac{B_z^2}{4\pi}\right\rangle - \langle \rho g_z \rangle = 0$$

Support by: gas pressure turbulence magnetic pressure

Confinement by: magnetic tension gravity

- For special case of gravity due to an infinite thin slab of surface density Σ:

$$\frac{d}{dz}\left\langle P + \rho v_z^2 + \frac{B^2}{8\pi}\right\rangle - \frac{d}{dz}\left\langle \frac{B_z^2}{4\pi}\right\rangle - 2\pi G \Sigma \langle \rho \rangle = 0$$

Approaches to the problem

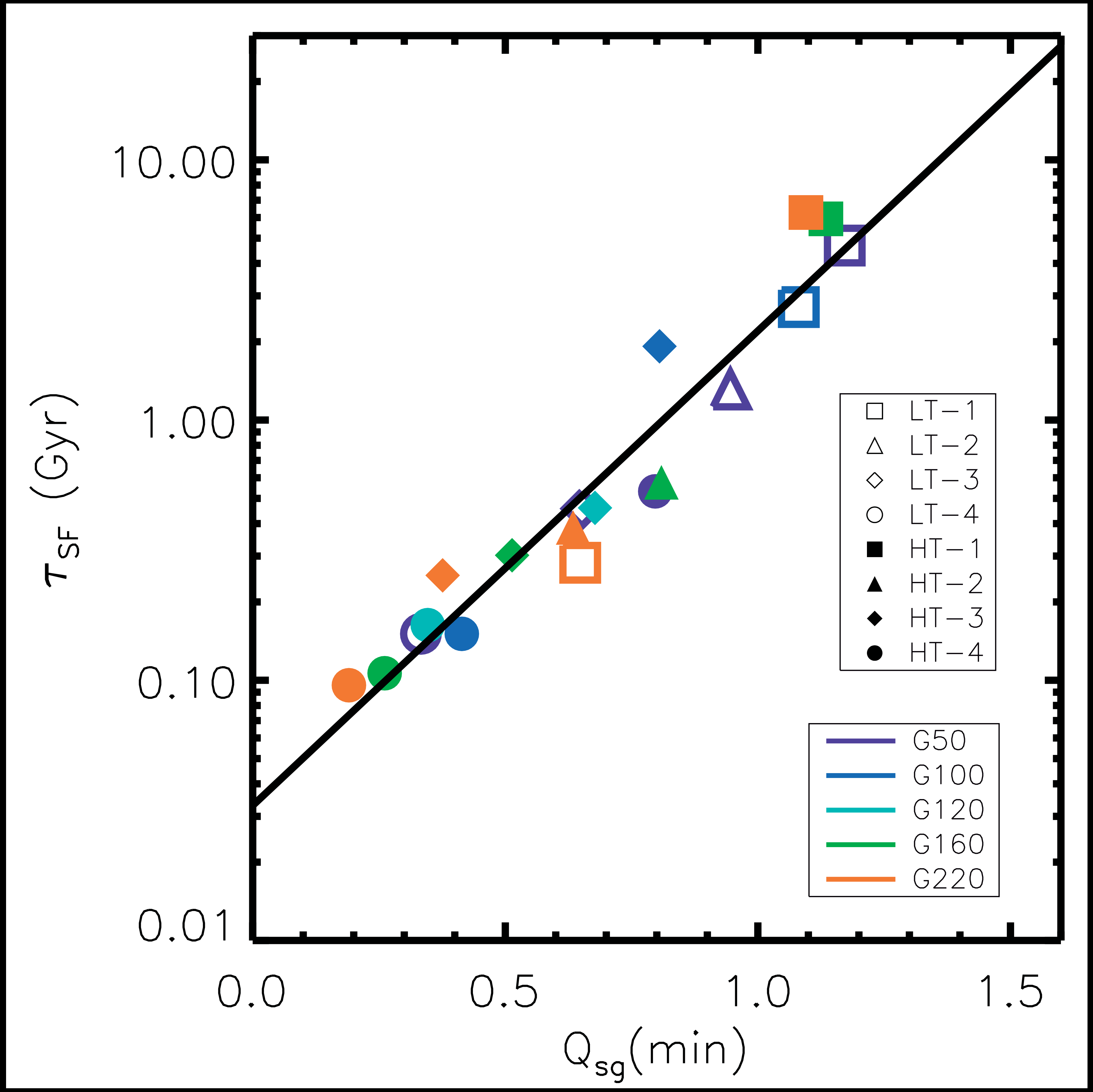
Top-down vs. bottom-up

- Goal is to explain observed correlation between galaxy properties (mass traced by CO, HCN, or HI, galaxy rotation curve, stars, etc.) and SFR
- Big questions to answer: (1) why is star formation so slow / inefficient? (2) why does star formation correlate with molecular phase?
- Two main approaches:
  - Top-down: attempt to model galactic disc as a whole, explain SFR based on properties of the disc; properties of individual GMCs maybe added later
  - Bottom-up: start from modelling formation of GMCs, try to understand SFR of them; build-up galaxy-scale star formation relation as sum of GMCs

Hydro + gravity only models

The baseline

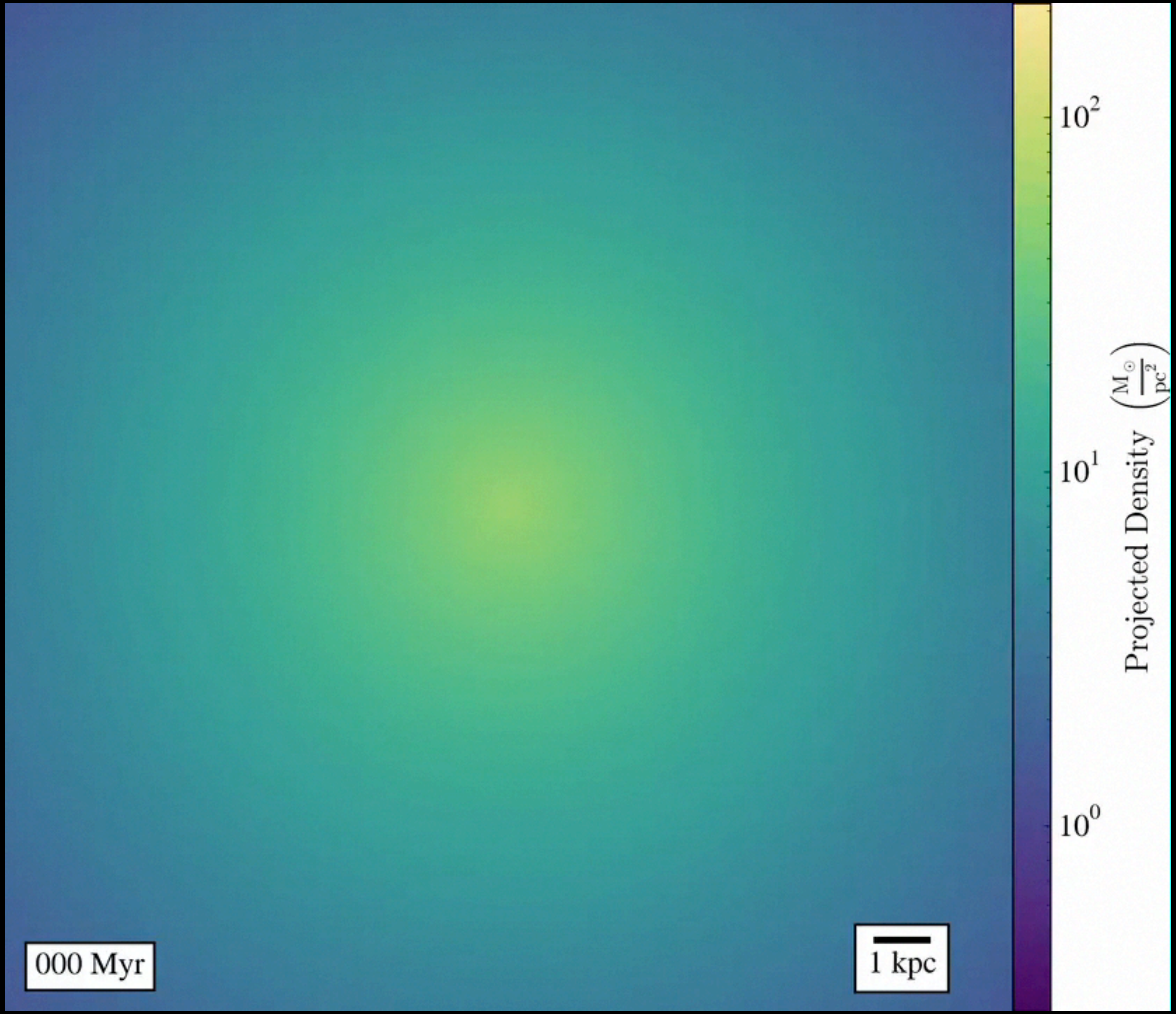
- Simplest case is isothermal gas, hydro + gravity only; no cooling, no feedback
- In this case, SFR correlated with *Q*; for high enough *Q*, can get low SFR
- However, this is artificial, because in absence of star formation, no mechanism exists to keep gas isothermal
- If gas can cool, initially high *Q* goes down in < 1 galactic orbit



Li+ 2005

Feedback!

- Most common solution to this problem: add feedback
- SN are dominant feedback; in galaxy-scale simulations, usually implemented by adding momentum directly to gas
- Simulations with feedback seem to produce reasonable SFRs for Milky Way-like galaxies



Goldbaum+ 2016

Feedback-regulated models

The basic idea

- Start from averaged, time-steady *z* momentum equation:

$$\frac{d}{dz}\left\langle P + \rho v_z^2 + \frac{B^2}{8\pi}\right\rangle - \frac{d}{dz}\left\langle \frac{B_z^2}{4\pi}\right\rangle - 2\pi G\Sigma \left\langle \rho \right\rangle = 0$$

- Assume magnetic tension term is small, and that turbulent + (magnetic) pressure term is set by balance between SF feedback and dissipation

Scale height

Velocity dispersion

- Dissipation rate / volume ~ (energy / volume) / crossing time: $\dot{E}_{\mathrm{diss}} \sim E/(H/\sigma)$
- Injection rate / volume ~ (SFR / volume) (momentum / SFR) (velocity scale) ~ $\dot{E}_{\mathrm{in}} \sim (\dot{\Sigma}_*/H) \left\langle p/M_* \right\rangle \sigma$
- Implied steady state: $\left\langle P + \rho v_z^2 + \frac{B^2}{8\pi}\right\rangle \sim E \sim \dot{\Sigma}_* \left\langle \frac{p}{M_*}\right\rangle$

Momentum / mass due to SNe ~ 3000 km/s/$M_\odot$ for isolated SNe

Feedback-regulated models

Implications

- Solve momentum balance equation for SFR / area:

$$\left\langle P + \rho v_z^2 + \frac{B^2}{8\pi} \right\rangle \sim 2\pi G \Sigma \langle \rho \rangle H$$

$$\dot{\Sigma}_* \sim 2\pi \left\langle \frac{p}{M_*} \right\rangle^{-1} G\Sigma^2$$

$$\sim 0.1\, M_\odot\ \mathrm{pc}^{-2}\ \mathrm{Myr}^{-1} \left(\frac{\Sigma}{100\, M_\odot\ \mathrm{pc}^{-2}} \right)$$

- Quantitatively, this is in the right ballpark compared to observations

- Model does not automatically enforce $Q \sim 1$, but can additionally hypothesise that scale height / volume density is set to the value required to satisfy this

Feedback-regulated models

Challenges and problems

- Predicted index of KS relation is 2 — too steep compared to observations, even with strong assumptions about α_{CO}
- Enforcing $Q = 1$ requires $t_{\mathrm{ff}} \sim 1/\Omega \rightarrow \rho \sim \Omega^2$, but with some algebra can show this implies that $\varepsilon_{\mathrm{ff}} \sim Q\sigma \rightarrow \varepsilon_{\mathrm{ff}}$ must vary with σ; also not observed
- In general no explanation for observed non-variation of $\varepsilon_{\mathrm{ff}}$; if SF is regulated by SN, why don't clouds that don't contain SN collapse at free-fall?
- No explanation for phase-dependence: if SFR just set by balance between ISM weight and SN momentum, why does gas phase matter at all?

Bottom-up models

Basic idea

- Step 1: figure out where in a galaxy there will be “GMCs” (i.e., gas that is capable of forming stars)
- Step 2: figure out star formation rate in the GMCs
- Build galaxy-scale star formation law out of these two pieces

Where can the ISM form stars?

And how is this related to ISM phase?

- Basic idea: only H_2 forms stars due to effects of shielding — H_2 represents phase of the ISM where gas is shielded from interstellar radiation field (ISRF)
- Heating vs. cooling balance in regions cooled by C^+:

Dust metallicity normalised to MW

Dust optical depth

MW unshielded FUV heating rate, ≈ 4 × 10^{-26} erg s^{-1}

$$\Gamma = \Gamma_{\mathrm{UV},0}\chi_{\mathrm{UV}} Z_d' e^{-\tau_d} + \Gamma_{\mathrm{CR}}\zeta'$$

CR ionisation rate normalised to MW

CR heating rate normalised to MW, ≈ 2 × 10^{-27} erg s^{-1}

ISFR strength normalised to MW

$$\Lambda = k_{\mathrm{de}} e^{-T_{\mathrm{C+}}/T} \delta_{\mathrm{C}} k_B T_{\mathrm{C+}} n_{\mathrm{H}}$$

Collisional de-excitation rate, ≈ 8 × 10^{-10} cm^3 s^{-1}

H number density

C abundance, ≈ 1.1 × 10^{-4} for MW

Temperature of cooling level, ≈ 91 K

- Solve for equality: $$T = -\frac{T_{\mathrm{C+}}}{\ln\left(\underbrace{0.36\chi_{\mathrm{UV}}e^{-\tau_d}}_{\text{UV heating term}} + \underbrace{0.018\zeta'/Z_d'}_{\text{CR heating term}}\right) - \ln\left(n_{\mathrm{H}}/100\ \mathrm{cm}^{-3}\right)}$$

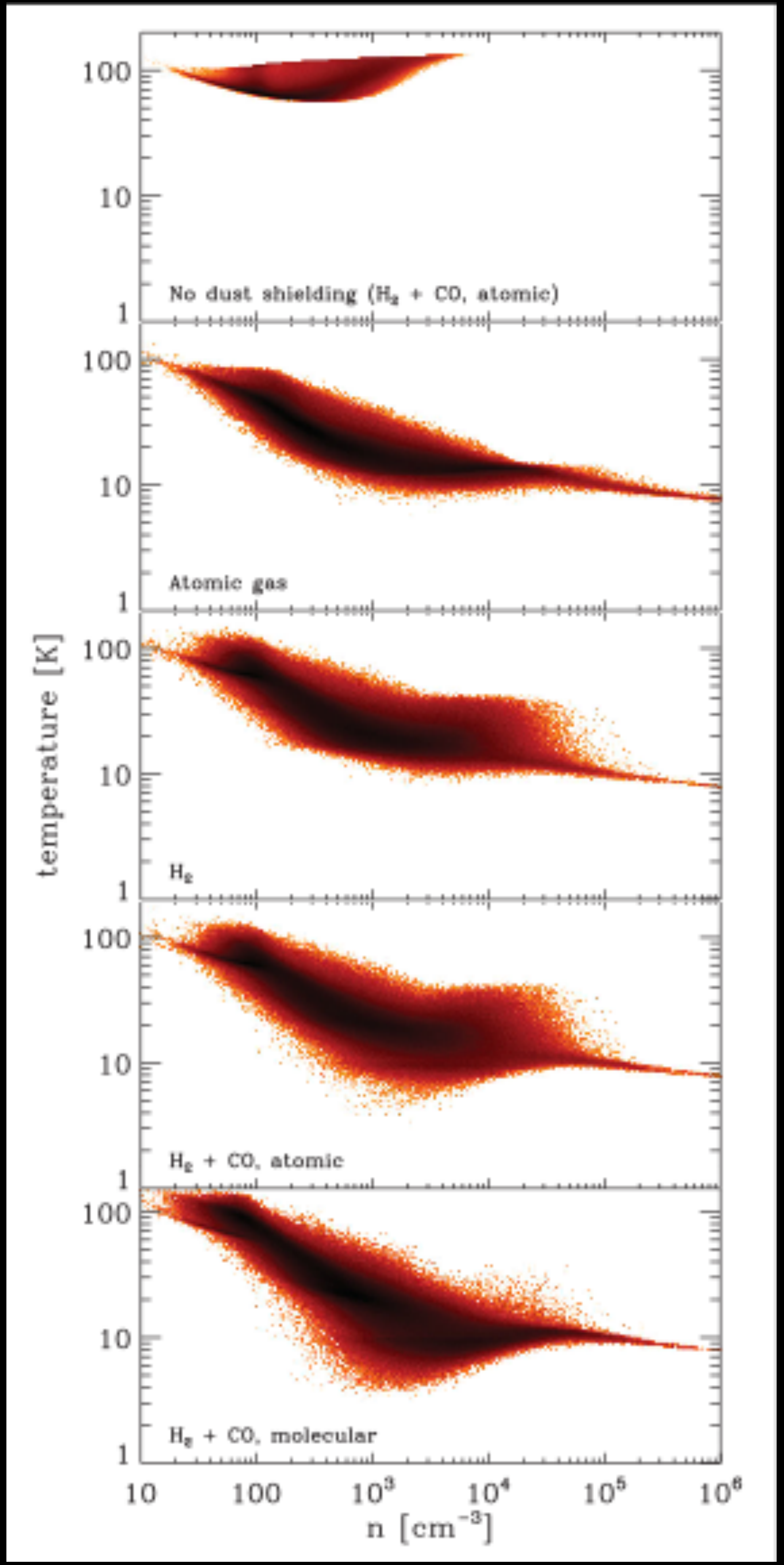
Why phase matters

CR vs. FUV heating

- Result depends on whether UV or CR heating term dominates; changeover occurs at τ_d ~ 3
- Temperatures at 100 cm^{-3}:

$$T \approx \frac{91\ \mathrm{K}}{1.0 + \tau_d - \ln \chi_{\mathrm{FUV}}} \quad \text{(UV)} \qquad T \approx \frac{23\ \mathrm{K}}{1.0 - 0.25 \ln(\zeta'/Z'_d)} \quad \text{(CR)}$$

- Factor of ~10 temperature difference → factor of 30 difference in Jeans mass — big difference in SFR
- Only form H_2 at high τ_d: this is why stars form in H_2, and why metallicity matters



Glover & Clark 2011

The SFR in GMCs

Why ε_{ff} is so small

- Consider turbulent medium with LWS relation $\sigma = c_s (\ell / \lambda_s)^{1/2}$
- Maximum mass that can be held up by thermal pressure: $M_J \sim \rho \lambda_J^3$; corresponding potential energy $\mathscr{W} \sim -G M_J^2 / \lambda_J \sim -(c_s^5 / G^3 \rho)^{1/2}$
- Compare to turbulent energy, evaluated using LWS: $\mathscr{T} \sim M_J \sigma^2 \sim -(\lambda_J / \lambda_s) \mathscr{W}$
- Key insight: in typical GMC, at mean density, $\lambda_J \gg \lambda_s$, so most gas is unbound
- Gas is bound only if density ≫ mean (→ $\lambda_J \ll$ mean)

Calculation of the critical density

- Mean-density Jeans length is $\lambda_{J,0} = (\pi c_s^2 / G\langle\rho\rangle)^{1/2}$, and at relative log density $s = \ln(\rho / \langle\rho\rangle)$, Jeans length is $\lambda_{J,0} = \lambda_{J,0}\, e^{-s/2}$
- Thus gas is bound for $s > s_{crit} \approx 2 \ln(\lambda_{J,0} / \lambda_s)$
- For cloud of mass M, radius R, sound speed c_s, $\lambda_{J,0} = 2\pi c_s (R^3 / 3GM)^{1/2}$, and $\lambda_s = 2R / \mathscr{M}^2$; combine to get $s_{crit} \approx \ln(\alpha_{vir}\mathscr{M}^2)$
- Estimate ε_{ff} ~ fraction of mass above s_{crit}:

$$\epsilon_{\rm ff} \sim \int_{s_{\rm crit}}^{\infty} p_M(s)\, ds = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{-2s_{\rm crit} + \sigma_s^2}{2^{3/2}\sigma_s}\right)\right]$$

- For α_{vir} ~ 2, $\mathscr{M}$ ~ 30, resulting ε_{ff} ~ 0.01, as required

Bottom-up models

Challenges and problems

- Bottom-up models do not address vertical force balance or Toomre *Q*; these need to be added (but can be, as paper we are reading shows)
- Turbulence alone only keeps ε_{ff} low as long as PDF is lognormal
  - However, simulations show that, in a self-gravitating medium, without feedback a power law tail develops that causes ε_{ff} to rise over time
  - Therefore, need local feedback process to keep low efficiency; exact nature of this local feedback (e.g., protostellac outflows) still not entirely clear