1. **Estimating α from observations. [30 points]**
Download the compilation of measurements of accreting stars and discs from the course web page. (These data are taken from Rafikov, 2017, ApJ, 837). This compilation contains stellar masses, luminosities, and accretion rates, along with the masses of the circumstellar discs. It does not contain disc radii, so we will assume that all discs have radii of 100 AU, typical of protostellar discs.

   (a) Using these data, estimate the range of kinematic viscosities ν in the observed discs. Plot the stellar accretion rate $\dot{M}_*$ versus the disc mass M_d, and show curves corresponding to a reasonable range of ν values in this plane.

   (b) In order to turn these estimates of viscosity into estimates of α, we require knowledge of the disc temperature. Assume that the temperature at the outer edge of the disc is set by the balance between heating by the central star and radiative cooling of spherical, blackbody dust grains, and that the disc is optically thin at its outer edge. Under these assumptions, derive an estimate for the temperature T at the outer edge of the disc; plot a histogram of T values.

   (c) Estimate α for these discs, and plot a histogram of $\log \alpha$ values. What are typical inferred α values?

2. **Steady gravity-dominated discs. [10 points]**
In this problem we will find a steady-state solution for steady discs that are marginally stable against self-gravity, $Q = 1$; such a configuration may be a reasonable description of discs around more massive stars during the main accretion phase. We consider an isothermal Keplerian disc with sound speed c_s orbiting a star of mass M_*.

   (a) Assuming that the disc maintains $Q = 1$ everywhere, compute the surface density as a function of radius.

   (b) Suppose that the disc has a constant radial inflow rate $\dot{M}$. Find the corresponding kinematic viscosity ν and dimensionless viscosity parameter α. (Hint: you will at some point encounter a constant of integration. Argue that you can drop it because it must become unimportant at large radii.)

3. **Entropy evolution of protostars. [25 points]**
One useful lens through which to consider low-mass protostellar evolution is by tracking the specific entropy of the stellar material – particularly because low-mass protostars are almost exactly $n = 3/2$ polytropes, which means they are isentropic. Recall that a polytrope is defined as a star that obeys a pressure-density relation $P = K\rho^{1+1/n}$ for some n, where K is called the polytropic constant.

   (a) Derive the relationship between K and the specific entropy s of the gas in a polytropic star, and show that for $n = 3/2$, s is independent of ρ or T. You may treat the stellar material as an ideal gas, and, as usual for entropies, you may leave an arbitrary additive constant in your answer.

   (b) Consider a protostar of mass M and radius R, and assume the structure is well-described by an $n = 3/2$ polytrope. Derive an expression for the specific entropy in terms of M and R.

(c) As the star contracts toward the main sequence along the Hayashi track, does the specific entropy of the star increase or decrease? Justify your answer.

(d) How does the specific entropy of a typical protostar of mass $M = 0.5\ M_\odot$, radius $R = R_\odot$ compare to the specific entropy of gas in a typical protostellar core (say $n = 10^6\ \mathrm{cm}^{-3}$, $T = 10$ K)? Is the star or the core a higher entropy state? For the purposes of this problem, you may ignore the difference between the chemical state of the core (molecular) and the star (fully ionised) – for simplicity just treat the core as fully ionised too, since it makes no difference to the qualitative result.

(e) You should find that the progression from core to protostar to main sequence star is one of decreasing specific entropy. However, the second law of thermodynamics says that total (rather than specific entropy) can never decrease, so if the gas in the star lost entropy, it must have gone somewhere else. Where did the entropy go?