1. **Magnetic fields of clouds and stars. [10 points]**
Consider a 1 $M_\odot$ protostellar core that collapses to make a star like the Sun. A typical observed density for such an object is 10^5 H_2 molecules cm^{-3}.

   (a) Suppose that such an object has an initial uniform magnetic field, and that it is governed by ideal MHD during its collapse. Make an order of magnitude estimate of how much larger the magnetic field of the resulting star is compared to the field of the initial core.

   (b) Consult the literature and look up typical observed magnetic field strengths for star-forming cores at a density of $\sim 10^5$ cm^{-3}, and for the surface magnetic fields of T Tauri stars. Two suggested papers to consult are Johns-Krull (2007, ApJ, 664, 975) and Crutcher (2012, ARA&A, 50, 29), but feel free to use others. Are the observed magnetic field strengths of cores and T Tauri stars consistent with the idea that the collapse is governed by ideal MHD? If not, by what factor must the magnetic field be increased or decreased?

2. **The singular isothermal sphere. [30 points]**
The singular isothermal sphere (SIS) is a simple model for the collapse of an initially-hydrostatic core to form a star. We consider a spherically-symmetric isothermal fluid with sound speed c_s, and work with the equations of mass and momentum conservation in terms of mass shells as derived in class:

$$\begin{aligned}
\frac{\partial M}{\partial r} &= 4\pi r^2 \rho \\
\frac{\partial M}{\partial t} + v\frac{\partial M}{\partial r} &= 0 \\
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} &= -\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial r} - \frac{GM}{r^2},
\end{aligned}$$

where M is the mass interior to radius r. Our goal is to find a solution that starts out as hydrostatic, but then develops a collapsing region that expands as time goes on.

   (a) The initial condition for the SIS is a configuration in which the gas is at rest ($v = 0$) and in force balance ($\partial v/\partial t = 0$), and the density varies with radius as $\rho \propto r^{-2}$. Show that such a configuration is a valid solution to the equation of momentum conservation, and derive the values of ρ and M as a function of r for this solution.

   (b) We will now look for a non-static similarity solution, which approaches the hydrostatic solution at time $t \to 0$. As a first step, we will make a similarity transformation by defining $x = r/c_s t$, $\rho = \alpha(x)/4\pi G t^2$, $M = (c_s^3 t/G)m(x)$, and $v = c_s u(x)$, where $\alpha(x)$, $m(x)$, and $u(x)$ are the dimensionless density, mass, and velocity. As a first step in this direction, use the first two equations to prove that $m = x^2\alpha(x-u)$. Hint: for quantities q, it is helpful to rewrite $\partial q/\partial t = (\partial x/\partial t)_r(\partial q/\partial x)_r = (-x/t)(\partial q/\partial x)_r$, and $\partial q/\partial r = (\partial x/\partial r)_t(\partial q/\partial x)_t = (x/r)(\partial q/\partial x)_r$, where the subscript r and t indicate that the partial derivative is to be evaluated at constant r or t.

   (c) Next use the second and third equations, together with your result from the previous part, to show that the non-dimensional equations describing the system can be written

$$\begin{aligned}\frac{1}{\alpha}\frac{d\alpha}{dx} &= \left[\frac{x-u}{(x-u)^2-1}\right]\left[\alpha-\frac{2}{x}(x-u)\right]\\ \frac{du}{dx} &= \left[\frac{x-u}{(x-u)^2-1}\right]\left[\alpha(x-u)-\frac{2}{x}\right].\end{aligned}$$

(d) We now want to find a solution that approaches the hydrostatic one at $x \gg 1$ (corresponding to large radius or early time), but that approaches a collapse solution as $x \to 0$. Since a finite amount of mass will have collapsed to the origin at any time > 0, such a solution must have $m \to m_0$ as $x \to 0$, where m_0 is some constant. To find the remainder of the solution, suppose that α and u have the limiting behaviours $\alpha \to \alpha_0 x^{-p}$ and $u \to -u_0 x^{-q}$ as $x \to 0$, where the powers p and q (both > 0) and the coefficients α_0 and u_0 (also both > 0) are to be determined by plugging into the equations. Solve for p, q, α_0, and u_0 in terms of m_0.

(e) The final step is to determine the value of m_0 numerically. We can find this by numerically integrating the equations given in part (c) inward from some starting $x \gg 1$, using $u = 0$ the value of α given by the hydrostatic solution found in part (a). This numerical integration should show that m approaches a constant value m_0 as $x \to 0$. Find m_0. (Note: depending on your numerical integration scheme, you may need to start u_0 off with a small but non-zero negative value to get the numerical integration to behave.)

(f) For this solution, what is the mass accretion rate onto the central object as a function of time. Express your answer in physical rather than dimensionless units.

3. **Photoionisation feedback in centrally-concentrated clouds. [20 points]**
In class we derived the solution describing the expansion of a photoionised region expanding into a uniform density background medium. In this problem we will generalise that solution to the case of a photoionised region that begins expanding from the centre of a cloud where the density as a function of radius is

$$\rho = \rho_e \left(\frac{r}{r_e}\right)^{-k},$$

where ρ_e and r_e are the density and radius at the edge of the core, and k is a constant. The gas is initially at rest, until it is swept up by the expanding bubble, which is driven by a source with ionising luminosity S.

(a) Argue that, even though the cloud outside the expanding bubble is centrally concentrated, the gas inside the photoionised region should quickly reach near-uniform density. (Hint: think about pressure inside the ionised bubble.)

(b) Use momentum conservation to derive an equation of motion for the swept-up shell; this should be analogous to equation 7.29 of the textbook.

(c) Find a late-time similarity solution to the equation you derived in the previous part, applicable once the shell has expanded to a radius much larger than the initial Strömgren radius.

(d) You should find that there is a critical value of k that separates solutions where the shell decelerates from those where it accelerates. Only the decelerating solutions are physically valid, since the accelerating solutions violate the assumption that the ionised bubble sweeps up a dense shell. Find the critical value of k, and give a physical interpretation for why this is the value that separates accelerating from decelerating solutions.