Class 18 Notes: H I: Vertical Distribution and Gravitational Stability

H I is the dominant phase of the ISM in the bulk of present-day galaxies. It is therefore useful to consider its large-scale disposition and organisation, particularly in the vertical direction. Now that we understand the phase structure of H I and its characteristic thermal behaviour, we are in a position to understand how it behaves in the potential well of a galaxy.

I. Hydrostatic balance

As a first step, let us consider the vertical distribution of the atomic ISM in a galactic disc. In the Milky Way and similar galaxies, some of the volume of the disc is filled with hot, ionised gas, and some (a tiny fraction) is filled with molecular gas, but somewhere between half and 2/3 of the volume is occupied by atomic gas. Due to its lower density, most of this volume (though not necessarily most of the mass) consists of gas in the warm phase. At the largest scale, this gas must be in approximate hydrostatic equilibrium, held up by its own pressure against both its own gravity and the gravity of the stars.

A. Infinite isothermal gas discs

We will start with the simplest possible case: a planar disc of gas surface density is Σ_g where the gas velocity dispersion is σ_g, independent of position. We imagine that σ_g is fixed by the radiative effects we discussed last class, which fix the warm gas temperature and sound speed. For now assume that the gravitational potential arises from the gas alone, with no contribution from stars, dark matter, bound gas clouds, or anything else. We further imagine that the disc is extremely large in extent in the horizontal direction, so we can treat the problem as simply an infinite slab.

With these approximations, the equation of hydrostatic balance reads

$$\frac{dp}{dz} = -\rho_g g, \tag{1}$$

where z is the direction normal to the disc plane, $p = \rho_g \sigma_g^2$ is the pressure, and

$$g = 4\pi G \int_0^z \rho_g \, dz \tag{2}$$

is the vertical gravitational acceleration. Integrating both sides we have

$$\rho_g(z) - \rho_g(0) = -\frac{1}{\sigma_g^2} \int_0^z \rho_g g \, dz. \tag{3}$$

The equation is most easily solved via a change of variables. We let

$$s_g = 2\int_0^z \rho_g \, dz \tag{4}$$

be the half-column density below vertical height z, so that $g = 2\pi G s_g$ and $ds_g = 2\rho_g \, dz$. With this change of variables, the integral on the right hand side simplifies to

$$\rho_g(z) - \rho_g(0) = -\frac{\pi G}{\sigma_g^2}\int_0^{s_g} s_g' \, ds_g' = -\frac{\pi G}{2\sigma_g^2} s_g^2 \tag{5}$$

Since we must have $\rho_g(z) \to 0$ and $s_g \to \Sigma_g$ as $z \to \infty$, the midplane density and pressure must be related to the total surface density and gas velocity dispersion by

$$\rho_g(0) = \frac{\pi}{2}\frac{G\Sigma_g^2}{\sigma_g^2} \qquad\qquad p(0) = \frac{\pi}{2} G\Sigma_g^2. \tag{6}$$

Thus we can write the gas density at the point where the half-column density is s as

$$\rho_g = \rho_g(0)\left[1 - \left(\frac{s_g}{\Sigma_g}\right)^2\right]. \tag{7}$$

We can also solve for the density as a function of height directly by plugging this into the definition of s:

$$s_g = 2\rho_g(0)\int_0^z \left[1 - \left(\frac{s_g}{\Sigma_g}\right)^2\right] dz \qquad \Longrightarrow \qquad \frac{ds_g}{dz} = 2\rho_g(0)\left[1 - \left(\frac{s_g}{\Sigma_g}\right)^2\right]. \tag{8}$$

This differential equation may be solved exactly; the solution is

$$s_g = \Sigma_g \tanh\left(\frac{z}{2h_g}\right) \qquad\qquad \rho_g = \rho_g(0)\,\mathrm{sech}^2\left(\frac{z}{2h_g}\right), \tag{9}$$

where

$$h_g = \frac{\Sigma_g}{\rho_g(0)} = \frac{\sigma_g^2}{2\pi G\Sigma_g} \tag{10}$$

is the gas scale height. We have therefore determined the full vertical density and pressure distribution for an isothermal gas disc.

B. Star plus gas disks

Now let us generalize our treatment to include stars. We take Σ_* and σ_* to be the stellar surface density and velocity dispersion, respectively. Generally we have $\Sigma_g \ll \Sigma_*$ and $\sigma_g \ll \sigma_*$ for Milky Way-like galaxies. For example, in the Solar neighborhood we have $\Sigma_g \approx 12\ M_\odot$ pc^{-2} (Boulares & Cox 1990), $\Sigma_* \approx 44\ M_\odot$ pc^{-2} (Holmberg & Flynn 2003), and $\sigma_g \approx 6$ km s^{-1} (Heiles & Troland 2003). The stellar velocity dispersion is a function of stellar age, and range from $\sigma_* \sim 40$

km s^{-1} for $\lesssim 1$ Gyr-old stars to ~ 80 km s^{-1} for many Gyr-old stars (Quillen & Garnett 2001).

For the gas, the equation of hydrostatic balance is exactly the same as before, except that now the gravitational force includes contributions from stars as well as gas:

$$g = 4\pi G \int_0^z (\rho_g + \rho_*)\, dz. \tag{11}$$

In the limit $\rho_g \gg \rho_*$, clearly we get back to the pure gas case we just solved. In the opposite limit, $\rho_* \gg \rho_g$, we can drop the ρ_g term, in effect neglecting the gas contribution to the gravitational potential. In this limit we can also assume that the stars have the density distribution expected of a pure stellar disc, with no gas, which is exactly the same as for an isothermal gas disc:

$$\rho_*(z) = \rho_*(0)\,\mathrm{sech}^2\left(\frac{z}{2h_*}\right), \tag{12}$$

with

$$h_* = \frac{\sigma_*^2}{2\pi G \Sigma_*} \qquad \rho_*(0) = \frac{\pi G \Sigma_*^2}{2\sigma_*^2}. \tag{13}$$

Substituting this in, it is trivial to obtain g by integration:

$$g = 4\pi G \rho_*(0)(2h_*)\tanh\left(\frac{z}{2h_*}\right) = 2\pi G \Sigma_* \tanh\left(\frac{z}{2h_*}\right). \tag{14}$$

Substituting this into the equation of hydrostatic balance we have

$$\sigma_g^2 \frac{d\rho_g}{dz} = -\rho_g\left[2\pi G \Sigma_* \tanh\left(\frac{z}{2h_*}\right)\right]. \tag{15}$$

This can obviously be integrated by separation of variables:

$$\frac{d\rho_g}{\rho_g} = -\frac{2\pi G\Sigma_*}{\sigma_g^2}\tanh\left(\frac{z}{2h_*}\right) dz. \tag{16}$$

The solution is

$$\rho_g = \rho_g(0)\left[\cosh\left(\frac{z}{2h_*}\right)\right]^{-2\sigma_*^2/\sigma_g^2}. \tag{17}$$

One can obtain the normalization constant by requiring that $2\int_0^\infty \rho_g\, dz = \Sigma_g$. Evaluating this gives a midplane density and pressure

$$\rho_g(0) = \sqrt{\pi} G \frac{\Sigma_g \Sigma_*}{\sigma_g \sigma_*} \qquad p(0) = \sqrt{\pi} G \Sigma_g \Sigma_* \frac{\sigma_g}{\sigma_*} \tag{18}$$

The pressure is σ_g^2 times this.

It is instructive to write down a formula that interpolates between the gas-dominated limit, where $p \propto G\Sigma_g^2$, and the star-dominated limit, where $p \propto G\Sigma_g\sigma_g/\sigma_*$. The approximate expression

$$p(0) \approx \frac{\pi}{2} G\Sigma_g \left(\Sigma_g + \Sigma_* \frac{\sigma_g}{\sigma_*}\right) \tag{19}$$

agrees with exact numerical solutions to within 10%, and is a useful rule of thumb. Thus we see that the cross-over between the gas-dominated and stellar-dominated regimes is determined by a comparison between Σ_g and $\Sigma_*(\sigma_g/\sigma_*)$. We may think of this second quantity as describing the total surface density of stellar material scaled by the relative scale heights of the gas and the star, or, equivalently, the surface density of stellar material within one *gas* scale height of the midplane. Stars that spend all their time well outside of the gas layer do not contribute to the gravitational force felt by the gas.

In the Milky Way, plugging in the values $\Sigma_g = 12\ M_\odot\ \mathrm{pc}^{-2}$, $\Sigma_* = 44\ M_\odot\ \mathrm{pc}^{-2}$, $\sigma_g = 6$ km s^{-1}, and $\sigma_* = 40$ km s^{-1} that we quoted earlier, we see that the combination $\Sigma_*(\sigma_g/\sigma_*) = 6.6\ M_\odot\ \mathrm{pc}^{-2}$. Since this is (somewhat) smaller than Σ_g, we conclude that the gas near the Solar circle is, at least marginally, dominated by self-gravity rather than stellar gravity, despite the fact that the surface density of stellar material exceeds that of gas by a factor of ~ 4. The difference is made up by the smaller velocity dispersion of the gas, so that only a relatively small fraction of the stellar material is near enough to the midplane at any time to exert a downard pull on the gas.

II. Toomre instability

Now that we have describe the hydrostatic "background state" of the gas, we can investigate some of the things that break that hydrostatic equilibrium. One obvious source of disturbance is feedback from star formation, in the form of H II regions (which we have discussed) and supernova blast waves (which we will discuss next week). However, even in the absence of these perturbations there are instabilities that can cause the gas to deviate from simply hydrostatic equilibrium.

The simplest instability in a thin disc is the Toomre instability, which is caused when the disc becomes self-gravitating. To work out the Toomre instability, consider a thin disc of H I in a galaxy. The gas has a constant velocity dispersion σ_g, and rotates about the galactic center with constant angular velocity Ω. We neglect stars, which is not a total unreasonable thing to do given that we have just shown that the vertical hydrostatic balance of the gas is, at least in some cases, dominated by gas, even if there is more mass in stars.

We set up a coordinate system so that the disc lies in the plane $z = 0$. We pick our origin to be at some point in the disc that is not at its center, and we set up coordinate system so that it is co-rotating with this point. The gas obeys the equations of mass and momentum conservation, plus the Poisson equation of self-gravity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{20}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\phi - 2\Omega\hat{\mathbf{e}}_z \times \mathbf{v} + \Omega^2(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y) \tag{21}$$

$$\nabla^2\phi = 4\pi G\rho, \tag{22}$$

where $\hat{\mathbf{e}}_{x,y,z}$ is the unit vector in the x, y, or z direction, and $p = \rho_c \sigma_g^2$ is the gas pressure. In the momentum equation, the last two terms represent the Coriolis force and the centrifugal force that arise in our rotating reference frame. Note here that $\mathbf{v}$ is in the rotating from as well, so $\mathbf{v} = 0$ represents gas that is smoothly rotating with the disc.

Our first step in describing the Toomre instability is to approximate that the disc is thin. We therefore write $\rho(z) = \Sigma\delta(z)$, and we take the z component of $\mathbf{v}$ to be zero. If we now integrate the first equation in the z direction and evaluate the second equation in the plane $z = 0$, we have

$$\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{v}) = 0 \tag{23}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla\Sigma}{\Sigma}\sigma_g^2 - \nabla\phi - 2\Omega\hat{\mathbf{e}}_z \times \mathbf{v} + \Omega^2(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y). \tag{24}$$

Now let us consider an equilibrium solution to these equations: $\Sigma = \Sigma_0$, $\mathbf{v} = 0$, and $\phi = \phi_0$. This represents a disc of gas in radial hydrostatic balance between the forces of gravity, pressure, and rotation. We wish to investigate whether this equilibrium is stable. To do so, we add a small perturbation to the surface density, velocity, and gravitational potential, and investigate how it evolves. We therefore let

$$\Sigma = \Sigma_0 + \epsilon\Sigma_1 \qquad \mathbf{v} = \epsilon\mathbf{v}_1 \qquad \phi = \phi_0 + \epsilon\phi_1, \tag{25}$$

where ϵ is small.

To determine how the perturbation evolves, we simply substitute the perturbed quantities into the equations of motion. For the continuity equation we obtain

$$\frac{\partial}{\partial t}(\Sigma_0 + \epsilon\Sigma_1) + \nabla \cdot (\epsilon\Sigma_0\mathbf{v}_1 + \epsilon^2\Sigma_1\mathbf{v}_1) = 0. \tag{26}$$

Since ϵ is small, we can drop the term of order ϵ^2, so we have

$$\frac{\partial \Sigma_1}{\partial t} + \nabla \cdot (\Sigma_0\mathbf{v}_1) = 0. \tag{27}$$

Doing the same trick with the momentum equation, we have

$$\epsilon\frac{\partial \mathbf{v}_1}{\partial t} + \epsilon^2(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 = -\frac{\nabla(\Sigma_0 + \epsilon\Sigma_1)}{\Sigma_0 + \epsilon\Sigma_1}\sigma_g^2 - \nabla(\phi_0 + \epsilon\phi_1) - \epsilon \cdot 2\Omega\hat{\mathbf{e}}_z \times \mathbf{v}_1 + \Omega^2(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y). \tag{28}$$

We can simplify this by dropping ϵ^2 terms and noting that $-\sigma_g^2(\nabla\Sigma_0/\Sigma_0) - \nabla\phi_0 + \Omega^2(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y)$ must vanish, since $\mathbf{v} = 0$ in the unperturbed state. Plugging this in and Taylor expanding in ϵ, we have

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\sigma_g^2 \frac{\nabla\Sigma_1}{\Sigma_0} - \nabla\phi_1 - 2\Omega\hat{\mathbf{e}}_z \times \mathbf{v}_1. \tag{29}$$

Finally, doing the same trick with the Poisson equation, we have

$$\nabla^2\phi_1 = 4\pi G\Sigma_1\delta(z). \tag{30}$$

These equations are easiest to solve by means of Fourier analysis. Since we can construct an arbitrary perturbation via a combination of single Fourier modes, there is no loss of generality in taking Σ_1 and $\mathbf{v}_1$ to be single Fourier modes. Thus we adopt

$$\Sigma_1 = \Sigma_a e^{i(kx-\omega t)} \qquad\qquad \mathbf{v}_1 = (v_{ax}\hat{\mathbf{e}}_x + v_{ay}\hat{\mathbf{e}}_y)e^{i(kx-\omega t)}. \tag{31}$$

Without loss of generality we can choose to orient our coordinate system so that the perturbation is in the x direction, so we need not consider a y component of $\mathbf{k}$.

For ϕ we need to know the behavior away from $z = 0$ as well as at $z = 0$. In order to satisfy the Poission equation, the behavior at $z = 0$ must also be a Fourier mode, so the solution must be of the form

$$\phi_1 = \phi_a e^{i(kx-\omega t)} f(z), \tag{32}$$

where $f(z)$ is some function of z that we define so that it is normalized to unity at $z = 0$. For $z \neq 0$ the density is zero, so to satisfy the Poission equation there we must have

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi_1 = \phi_a\left[-k^2 f(z) + f''(z)\right]e^{i(kx-\omega t)}. \tag{33}$$

Thus we have

$$f''(z) - k^2 f(z) = 0, \tag{34}$$

with the boundary conditions that $f(0) = 1$, and $f(z) \to 0$ as $z \to \pm\infty$, i.e. the potential vanishes far from the disc. Integrating the differential equation with these boundary conditions gives

$$f(z) = e^{-|kz|}, \tag{35}$$

so

$$\phi_1 = \phi_a e^{i(kx-\omega t)-|kz|}. \tag{36}$$

Note that f' (and thus ϕ_1' are undefined at $z = 0$. This is to be expected, since we have hypothesized that there is mass sheet of infinite thinness, and thus infinite density, at $z = 0$.

Now we must solve for ϕ_a in terms of Σ_a: physically, this means that we must figure out what sort of perturbation to the gravitational potential is created by the density perturbation we are imposing. To do this, it is helpful to integrate the Poisson equation over a small range in z to eliminate the δ function. If we integrate both sides from $z = -\tau$ to $z = \tau$, we have

$$\int_{-\tau}^{\tau} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi_1 \, dz = 4\pi G \Sigma_1 \int_{-\tau}^{\tau} \delta(z) \, dz = 4\pi G \Sigma_1. \tag{37}$$

Of the terms on the left hand side, $\partial^2 \phi_1 / \partial y^2 = 0$ since ϕ_1 does not depend on y, and we can make $\int_{-\tau}^{\tau} (\partial^2 \phi_1 / \partial x^2) \, dz$ arbitrarily small by choosing τ to be very small. We can therefore drop this term as well. This leaves only the z component:

$$\int_{-\tau}^{\tau} \frac{\partial^2 \phi_1}{\partial z^2} \, dz = \left(\frac{\partial \phi_1}{\partial z} \right)_{\tau} + \left(\frac{\partial \phi_1}{\partial z} \right)_{-\tau} = -2\phi_a |k|. \tag{38}$$

Therefore we have

$$\phi_a = -\frac{2\pi G \Sigma_a}{|k|}. \tag{39}$$

We are now ready to solve. We substitute our Fourier modes for Σ_1, $\mathbf{v}_1$, and ϕ_1 into the continuity equation and the momentum equation. After simplifying, we obtain

$$-i\omega \Sigma_a + ik\Sigma_0 v_{ax} = 0 \tag{40}$$

$$ik \left(\frac{\sigma_g^2}{\Sigma_0} - \frac{2\pi G \Sigma_a}{|k|} \right) \Sigma_a - i\omega v_{ax} - 2\Omega v_{ay} = 0 \tag{41}$$

$$2\Omega v_{ax} - i\omega v_{ay} = 0. \tag{42}$$

The first equation is from the continuity equation, and the next two are the x and y components of the momentum equation.

This represents three equations in the three unknowns Σ_a, v_{ax}, and v_{ay}. It is easiest to write them in matrix form:

$$\begin{pmatrix} -i\omega & ik\Sigma_0 & 0 \\ ik \left(\frac{\sigma_g^2}{\Sigma_0} - \frac{2\pi G \Sigma_a}{|k|} \right) & -i\omega & -2\Omega \\ 0 & 2\Omega & -i\omega \end{pmatrix} \begin{pmatrix} \Sigma_a \\ v_{ax} \\ v_{ay} \end{pmatrix} = 0. \tag{43}$$

Non-trivial solutions exist if and only if the determinant of the matrix is zero. Setting the determinant to zero and skipping a bunch of algebraic simplification, we obtain

$$\omega^2 = 4\Omega^2 + \sigma_g^2 k^2 - 2\pi G \Sigma_0 |k|. \tag{44}$$

This equation is a dispersion relation. It tells us that if we impose a perturbation with spatial frequency k at time $t = 0$, the system will respond with temporal frequency ω. There are two possibilities to consider. If the quantity $4\Omega^2 + \sigma_g^2 k^2 - 2\pi G \Sigma_0 |k|$ is positive, then ω is plus or minus a real number. Since the temporal behavior follows $e^{-i\omega t}$, this means that the system simply oscillates, and the magnitude of the perturbation stays the same. We refer to this as stability. On the other hand if $4\Omega^2 + \sigma_g^2 k^2 - 2\pi G \Sigma_0 |k|$ is negative, then ω is plus or minus an imaginary number, and the solution consists of one exponentially decaying mode and one exponentially growing mode. This is unstable, since a small perturbation will grow exponentially until it becomes large.

The condition for instability in a disc, known as Toomre's condition, simply that there exist a value of k for which

$$4\Omega^2 + \sigma_g^2 k^2 - 2\pi G \Sigma_0 |k| < 0. \tag{45}$$

By taking the derivative with respect to k, we see that the quantity on the right hand side reaches its minimum at $k = \pi G \Sigma_0 / \sigma_g^2$. Plugging in this value of k, we have

$$4\Omega^2 - \frac{\pi^2 G^2 \Sigma_0^2}{\sigma_g^2} < 0 \qquad \Longrightarrow \qquad \frac{2\sigma_g \Omega}{\pi G \Sigma_0} < 1. \tag{46}$$

We define

$$Q = \frac{2\sigma_g \Omega}{\pi G \Sigma_0} \tag{47}$$

as the Toomre Q parameter of a disc.

discs with $Q < 1$ are unstable to runaway gravitational collapse. The physical interpretation of Q is straightforward. The numerator represents the combined stabilizing effects of pressure (σ_g) and rotation (Ω), while the denominator represents the destablizing effects of gravity ($G\Sigma_0$). Larger surface densities tend to make the disc more unstable, while larger velocity dispersions or rotation rates make it more stable.

More general Q parameters can be defined for discs with non-constant angular velocity, and the only difference that introduces is that the factor 2 changes to a different numerical value. Q can also be defined for collisionless stellar discs, and for combined gas plus star discs.