# Class 10 Notes: Collisionally-excited lines

Now that we have understood something about how ionisation equilibrium is established, we will next consider processes that cause observable emission ionised gas. We have already encountered these processes from hydrogen atoms: free-free and recombination line emission. Our next goal is to extend that analysis to other elements. These are mostly excited by collisions, and thus the topic to which we turn is collisional excitation and collisionally excited line emission. This process is one of our best ways of learning about the properties of the ISM, and a clever use of collisional excitation tracers lets us infer temperatures, densities, chemical compositions, and a host of other properties in emitting regions. The key insight is that understanding when lines will be in LTE and when they will not, and seeing how far out of LTE they get, provides a powerful tool.

I. The two-level atom

We begin by considering the simplest case of an atom that has only two relevant states, which we will denote 0 and 1, with energies  $E_0$  and  $E_1$ ,  $E_0 < E_1$ . The number density of atoms in each state is  $n_0$  and  $n_1$ . Transitions between the two states happen in two ways. First, there is spontaneous emission from atoms in state 1, which occurs at a rate per unit volume  $A_{10}n_1$ , where  $A_{10}$  is the standard Einstein A coefficient. Collisions between the atoms in question and some collision partner – most often free electrons – with number density  $n_c$  can cause changes of state. The rate coefficient for transitions from state 0 to state 1 is  $k_{01}$ , and the rate coefficient for the reverse reaction is  $k_{10}$ .

A. The collision-only case

We will assume for the moment that there is no ambient radiation field (J = 0), so there is no absorption or stimulated emission; we will drop that assumption in a moment. Second,

Given this setup, we can compute the rate of change of the number density of particles in state 1:

$$\frac{dn_1}{dt} = n_c n_0 k_{01} - n_c n_1 k_{10} - n_1 A_{10}.$$
(1)

The steady-state solution,  $dn_1/dt = 0$ , is

$$\frac{n_1}{n_0} = \frac{n_c k_{01}}{n_c k_{10} + A_{10}}.$$
(2)

Now recall that  $k_{01}$  and  $k_{10}$  are not independent, since they are the backward and forward rate coefficients for the same reaction. Detailed balance requires that

$$k_{01} = \frac{g_1}{g_0} k_{10} e^{-E_{10}/kT_K} \tag{3}$$

where  $g_0$  and  $g_1$  are the degeneracies of the two states,  $E_{10} = E_1 - E_0$ , and  $T_K$  is the gas kinetic temperature – the temperature that describes the velocity distribution of the atoms. Note that we have not assumed LTE, we have only assumed that the atoms in question and their collision partners follow a Maxwellian velocity distribution, which is a much weaker assumption.

Substituting in for  $k_{01}$ , we can rewrite the steady-state solution as

$$\frac{n_1}{n_0} = \left[\frac{1}{1 + A_{10}/(n_c k_{10})}\right] \frac{g_1}{g_0} e^{-E_{10}/kT_K}.$$
(4)

It is clear from this expression that  $A_{10}/k_{10}$  has the units of density, so we define the critical density as

$$n_{\rm crit} = \frac{A_{10}}{k_{10}}.$$
 (5)

With this definition, we can rewrite the steady state solution as

$$\frac{n_1}{n_0} = \left(\frac{1}{1 + n_{\text{crit}}/n_c}\right) \frac{g_1}{g_0} e^{-E_{10}/kT_K}.$$
(6)

It is instructive to phrase this result in terms of the excitation temperature. Recall that we define the excitation temperature as the temperature that would be required to make the ratio of level populations follow the Boltzmann distribution, i.e., we define  $T_{\rm ex}$  implicitly by

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-E_{10}/kT_{\rm ex}}.$$
(7)

If we now equate this expression with our expression for the actual ratio  $n_1/n_0$ , we can solve for the excitation temperature:

$$T_{\rm ex} = \frac{T_K}{1 + (kT_K/E_{10})\ln(1 + n_{\rm crit}/n_c)}.$$
(8)

Now notice how this behaves in the limits  $n_c \gg n_{\text{crit}}$  and  $n_c \ll n_{\text{crit}}$ . In the first case,  $n_c \gg n_{\text{crit}}$ , the argument of the logarithm is close to unity, and thus the logarithmic term goes to zero. We therefore have  $T_{\text{ex}} \rightarrow T_K$ . In this case the system is in LTE.

In the second case,  $n_c \ll n_{\rm crit}$ , the logarithmic term is non-negligible, and thus  $T_{\rm ex} < T_K$ , with the limiting behaviour  $T_{\rm ex} \propto 1/\ln(n_{\rm crit}/n_c)$ . In terms of level populations, this corresponds to  $n_1/n_0 \propto n_c/n_{\rm crit}$ . Physically, what is going on in the regime  $n_c \ll n_{\rm crit}$  is that radiative de-excitation is depopulating the upper state faster than collisions can repopulate it, and the gas is therefore less excited than it would be in LTE – a situation called sub-thermal excitation. The larger the value of  $A_{10}$ , the faster this process happens, and the larger the density must be to keep up, which is why  $n_{\rm crit}$  is proportional to  $A_{10}$ .

B. Effect of an external radiation field

Now lets remove the assumption that there is no ambient radiation field, and suppose that there is one with directionally-averaged photon occupation number  $\langle n_{\gamma} \rangle$  at the frequency of the line. Including the terms for stimulated emission and absorption, the rate of change of atoms in state 1 becomes

$$\frac{dn_1}{dt} = n_0 \left( n_c k_{01} + \langle n_\gamma \rangle \frac{g_1}{g_0} A_{10} \right) - n_1 \left[ n_c k_{10} + (1 + \langle n_\gamma \rangle) A_{10} \right].$$
(9)

The first of the terms proportional to  $\langle n_{\gamma} \rangle$  represents the rate at which absorptions move atoms from state 0 to state 1, and the second represents the rate at which stimulated emissions depopulate state 1. The steady-state solution,  $dn_1/dt = 0$ , is now

$$\frac{n_1}{n_0} = \frac{n_c k_{01} + \langle n_\gamma \rangle (g_1/g_0) A_{10}}{n_c k_{10} + (1 + \langle n_\gamma \rangle) A_{10}}.$$
(10)

We can simplify this by using detailed balance, Equation 3, and defining a generalised critical density,

$$n_{\rm crit} = \frac{(1 + \langle n_\gamma \rangle) A_{10}}{k_{10}}.$$
(11)

Note that this is exactly the same as our previous definition, except that we have multiplied by a factor  $1 + \langle n_{\gamma} \rangle$ . With this definition, and a little bit of algebraic manipulation, we can rewrite the solution as

$$\frac{n_1}{n_0} = \left(\frac{1}{1 + n_{\text{crit}}/n_c}\right) \frac{g_1}{g_0} e^{-E_{10}/kT_K} + \left(\frac{1}{1 + n_c/n_{\text{crit}}}\right) \frac{\langle n_\gamma \rangle}{1 + \langle n_\gamma \rangle}.$$
 (12)

The first term is identical to what we got in the case without an external radiation field, while the second term is new.

To understand what this result means, it is helpful to rewrite  $\langle n_{\gamma} \rangle$  in terms of the brightness temperature  $T_B$ , which, we recall, is defined analogously to the excitation temperature: it is the temperature that would be required in LTE to produce a radiation field with this value of  $\langle n_{\gamma} \rangle$  at this frequency. This is given implicitly by

$$\langle n_{\gamma} \rangle = \frac{1}{e^{E_{10}/kT_B} - 1} \tag{13}$$

Substituting this in, the steady-state solution is

$$\frac{n_1}{n_0} = \left(\frac{1}{1 + n_{\text{crit}}/n_c}\right) \frac{g_1}{g_0} e^{-E_{10}/kT_K} + \left(\frac{1}{1 + n_c/n_{\text{crit}}}\right) \frac{g_1}{g_0} e^{-E_{10}/kT_B}.$$
 (14)

This is now completely parallel between kinetic and brightness temperature. Again, it is useful to look at limiting cases. For  $n_c \gg n_{\rm crit}$ , the first term in parentheses is big and the second one is small, so we can drop the radiation term involving  $T_B$ , and we just get the kinetic term involving  $T_K$ . This is the case we have already explored: for densities far above the critical density, the system goes into collisional equilibrium, with  $T_{\text{ex}} \to T_K$ .

In the opposite limit,  $n_c \ll n_{\rm crit}$ , the second term is big and the first is small. Since the functional form is identical to the case we have already analysed, we can immediately see the answer: in this case,  $T_{\rm ex} \to T_B$ . Thus the system goes to a different sort of equilibrium: instead of the excitation temperature going to the gas kinetic temperature, it goes to the radiation brightness temperature.

Finally, at  $n_c \sim n_{\rm crit}$ , we expect an intermediate result, where the excitation temperature is in between the kinetic and radiation temperatures. Thus the value of  $n_{\rm crit}$  tells us whether we should think of the system as collision-dominated, or radiation-dominated.

# C. Application I: the H I 21 cm line

Although two-level systems are relatively rare in nature, there are some that come close. The most obvious example is the 21 cm line of atomic hydrogen, where the two hyperfine spin states are separated by energy  $E_{10}/k = 0.0682$  K, and  $A_{10} = 2.88 \times 10^{-15} \text{ s}^{-1}$ . The collision rate coefficient for collisional de-excitation by collisions between H atoms at T = 100 K, a typical CNM temperature, is  $k_{10} \approx 1.2 \times 10^{-10} \text{ cm}^3 \text{ s}^{-1}$ ; a fit for other temperatures is given in *Draine*.

Let us calculate the equilibrium spin temperature of a cloud of atomic hydrogen as a function of its density at this temperature, assuming a radiation environment like that found in the Milky Way. The first step is to compute the critical density, and for this we must know  $\langle n_{\gamma} \rangle$ . At the frequency of the 21 cm line, the dominant sources of ambient radiation are the CMB, which provides  $T_B = 2.73$  K, and galactic synchrotron emission, which has  $T_B \approx 1$  K. Since we are in the Rayleigh-Jeans tail of the Planck function, the combined brightness temperature from both sources is simply their sum, so  $T_B = 3.73$  K. Plugging in,

$$\langle n_{\gamma} \rangle = \frac{1}{e^{E_{10}/kT} - 1} \approx 55. \tag{15}$$

Thus we have

$$n_{\rm crit} = \frac{(1 + \langle n_{\gamma} \rangle) A_{10}}{k_{10}} \approx 1.7 \times 10^{-3} \,{\rm cm}^{-3}.$$
 (16)

Pretty much any gas where the hydrogen is mostly H I will have a density well above this value, so in the present-day Universe, we expect the spin temperature in the CNM to be close to the gas kinetic temperature, rather than to the CMB plus synchrotron temperature. Note, however, that – per our earlier discussion – this was not necessarily the case in the early Universe, when the CMB was much more intense.

# D. Application II: the C II 158 $\mu$ m line

Another important application is the 158  $\mu$ m line of C II, which corresponds to the  ${}^{2}P_{3/2}^{o} \rightarrow {}^{2}P_{1/2}^{o}$  forbidden fine structure line. This turns out to be one of

the dominant cooling lines in the atomic ISM. The Einstein A for the line is  $A_{10} = 2.4 \times 10^{-6} \text{ s}^{-1}$ , a small value appropriate for a forbidden line. The two states are coupled by collisions with both neutral hydrogen atoms and the (small) population of free electrons found in predominantly neutral regions. The rate coefficients for collisional de-excitation are approximately

$$k_{10}(e^{-}) = 4.5 \times 10^{-8} (T/10^4 \text{ K})^{-1/2} \text{ cm}^3 \text{ s}^{-1}$$
 (17)

$$k_{10}(\mathrm{H}) = 7.6 \times 10^{-10} (T/10^2 \mathrm{K})^{0.13} \mathrm{cm}^3 \mathrm{s}^{-1}.$$
 (18)

Note that the charged-charged reaction has a rate coefficient that scales as  $T^{-1/2}$ , and the charged-neutral one has a rate coefficient that is nearly independent of temperature, as expected.

At 158  $\mu$ m, there is negligible background radiation – the CMB gives  $\langle n_{\gamma} \rangle \approx 10^{-5}$ . Thus we can consider collisions alone, and the critical densities for electrons and neutral hydrogen atoms are

$$n_{\rm crit}(e^-) \approx 53 (T/10^4 \text{ K})^{1/2} \text{ cm}^{-3}$$
 (19)

$$n_{\rm crit}({\rm H}) \approx 3.2 \times 10^6 (T/10^2 {\rm K})^{-0.13} {\rm cm}^{-3}.$$
 (20)

These values are much less than those typically found in the atomic ISM, so we conclude that the C II fine structure levels are sub-thermally excited.

#### II. Many-level atoms

Having developed a theory for the simplest case of two quantum states, we are now in a position to generalise to a system with an arbitrary number of quantum states. In practice such cases will almost always have to be solved numerically, but our goal here is to develop the models that can be used to do so.

# A. Formalism

Suppose we have a particle with states  $i = 0, 1, 2, \ldots$ , each with its own energy  $E_i$ and degeneracy  $g_i$ . Let  $E_{ij}$  be the difference in energy between any two states,  $A_{ij}$ be the rate of spontaneous emission between two states (where by convention we take  $A_{ij} = 0$  for any  $i \leq j$ , i.e., rates of "spontaneous excitation" are zero), and  $k_{ij}$  be the collision rate coefficient connecting the two states (and by convention  $k_{ii} = 0$ , i.e., we do not consider transitions from a state to itself). The number density of collision partners causing collisional excitations is again  $n_c$ .

First consider the case where there is no ambient radiation field. In this case, the rate of change of the number density of atoms in state i is then given by

$$\frac{dn_i}{dt} = \sum_j n_j \left( n_c k_{ji} + A_{ji} \right) - n_i \sum_j \left( n_c k_{ij} + A_{ij} \right).$$
(21)

The first sum represents the rate of transitions into state i from all other states j, and this has two parts: collisional transitions into state i, and spontaneous

radiative decays into state i from higher states. The second sums represents the rate of transitions out of state i summed over all states j into which the atom could go, and this again has two parts: collisions from i to j, and spontaneous decays from i to j.

Now let's add back in the complication of an ambient radiation field. This gives us two additional processes to worry about: stimulated emission and absorption. Stimulated emission is easy to handle: we just multiply the spontaneous emission rate by a factor of  $(1 + \langle n_{\gamma,ij} \rangle)$ , where  $\langle n_{\gamma,ij} \rangle$  is the photon occupation number, evaluated at the frequency  $\nu_{ij} = E_{ij}/h$  that corresponds to the energy difference between states *i* and *j*. Thus including stimulated emission but not absorption we have,

$$\frac{dn_i}{dt} = \sum_j n_j \left[ n_c k_{ji} + \left( 1 + \langle n_{\gamma,ij} \rangle \right) A_{ji} \right] - n_i \sum_j \left[ n_c k_{ij} + \left( 1 + \langle n_{\gamma,ij} \rangle \right) A_{ij} \right].$$
(22)

Finally, to include absorption, recall that the absorption rate from state  $i \to j$  is  $(g_i/g_j)n_i\langle n_{\gamma,ij}\rangle A_{ji}$ . Adding this term to both the rate of particles entering state i and leading state i, we have

$$\frac{dn_i}{dt} = \sum_j n_j \left[ n_c k_{ji} + \left( 1 + \langle n_{\gamma,ij} \rangle \right) A_{ji} \right] \sum_j n_g \frac{g_i}{g_j} \langle n_{\gamma,ij} \rangle A_{ij} 
- n_i \sum_j \left[ n_c k_{ij} + \left( 1 + \langle n_{\gamma,ij} \rangle \right) A_{ij} \right] - n_i \sum_j \frac{g_j}{g_i} \langle n_{\gamma,ji} \rangle A_{ji}.$$
(23)

The two new sums we have added are, for the positive term, the rate at which absorptions out of state j < i populate state i, and the rate at which absorptions out of state i into state j > i depopulate state i.

To find the steady state, we set  $dn_i/dt = 0$ , and we now need to solve a system of linear equations

$$\sum_{j} n_{j} \left[ n_{c} k_{ji} + (1 + \langle n_{\gamma,ij} \rangle) A_{ji} + \frac{g_{i}}{g_{j}} \langle n_{\gamma,ji} \rangle A_{ij} \right]$$
  
=  $n_{i} \sum_{j} \left[ n_{c} k_{ij} + (1 + \langle n_{\gamma,ik} \rangle) A_{ij} + \frac{g_{j}}{g_{i}} \langle n_{\gamma,ij} \rangle A_{ji} \right].$  (24)

This looks a bit daunting, but is somewhat easier to make sense of if we isolate the  $n_i$ . Re-arranging a bit, we have

$$n_i = \phi_i^{-1} \sum_j \left[ n_c k_{ji} + \left( 1 + \langle n_{\gamma,ij} \rangle A_{ji} + \frac{g_i}{g_j} \langle n_{\gamma,ij} \rangle A_{ij} \right] n_j,$$
(25)

where  $\phi_i$  is a normalisation factor, given by

$$\phi_i = \sum_j \left[ n_c k_{ij} + \left( 1 + \langle n_{\gamma,ij} \rangle A_{ij} + \frac{g_j}{g_i} \langle n_{\gamma,ij} \rangle A_{ji} \right].$$
(26)

We can recognise this as a matrix multiplication problem. Let  $\mathbf{n} = (n_0, n_1, n_2, ...)$  be the vector of level populations. In this case, the equation we have just written down is simply

$$\mathsf{M} \cdot \mathbf{n} = \mathbf{n},\tag{27}$$

where the matrix M has elements given by

$$M_{ij} = \phi_i^{-1} \sum_j \left[ n_c k_{ji} + \left( 1 + \langle n_{\gamma,ij} \rangle A_{ji} + \frac{g_i}{g_j} \langle n_{\gamma,ij} \rangle A_{ij} \right].$$
(28)

This is a standard eigenvalue problem, and the solution n is simply the eigenvector that corresponds to an eigenvalue of 1. The matrix M itself depends only on the known rate coefficients and the ambient radiation field and number density of collision partners. Thus the problem of finding the steady state number densities **n** simply reduces to the problem of solving a simple eigenvalue problem, which can be done by any of a number of standard techniques.

B. Critical densities for multi-level atoms

While this formalism can be used to solve multi-level atoms easily on a computer, we can gain some insight into how the solution is going to look if we define a generalised critical density for our multi-level atom. For the sake of simplicity, and to gain intuition, let us assume that there is a particular level i which is the highest energy level with any appreciable population, so that neither collisional de-excitations from even higher energy states, nor radiative transitions into our out of these higher energy states, contribute appreciably to its population. In practice this is almost always the case. For this level i, the rate equation becomes

$$\frac{dn_i}{dt} = \sum_{j < i} n_j n_c k_{ji} + \sum_{j < i} \frac{g_i}{g_j} \langle n_{\gamma, ij} \rangle A_{ij} - n_i \left[ n_c k_{ij} + \left( 1 + \langle n_{\gamma, ij} \rangle \right) A_{ij} \right].$$
(29)

The remaining terms describe, from left to right, collisional excitation from lower levels j into level i, radiative absorptions from level j into level i, and collisional and radiative de-excitations from level i to all other, lower levels j. The steady-state solution is

$$n_i = \frac{\sum_{j < i} n_j n_c k_{ji} + \sum_{j < i} n_j \frac{g_i}{g_j} \langle n_{\gamma,ij} \rangle A_{ij}}{\sum_{j < i} [n_c k_{ij} + (1 + \langle n_{\gamma,ij} \rangle) A_{ij}]}.$$
(30)

In analogy with the two-level case, we can define the critical density as the ratio of the radiative de-excitation rate coefficient to the collisional de-excitation rate coefficient,

$$n_{\operatorname{crit},i} = \frac{\sum_{j < i} \left(1 + \langle n_{\gamma,ij} \rangle\right) A_{ij}}{\sum_{j < i} k_{ij}}.$$
(31)

Using this definition, with a bit of algebra we can re-arrange the steady-state solution into something very similar to what we had for the two-level atom:

$$n_{i} = \left(\frac{1}{1 + n_{\operatorname{crit},i}/n_{c}}\right) \frac{\sum_{j < i} n_{j} k_{ji}}{\sum_{j < i} k_{ij}} + \left(\frac{1}{1 + n_{c}/n_{\operatorname{crit},i}}\right) \frac{\sum_{j < i} n_{j} \frac{g_{i}}{g_{j}} \langle n_{\gamma,ij} \rangle A_{ij}}{\sum_{j < i} \left(1 + \langle n_{\gamma,ij} \rangle\right) A_{ij}}.$$
 (32)

Thus we can see that  $n_{\rm crit}$  plays the same role here as in the two-level atom: if  $n_c \ll n_{\rm crit}$ , then the term describing collisional interactions is small, and the solution is entirely determined by radiative processes. If  $n_c \gg n_{\rm crit}$ , then the radiative term is small and the collision rate coefficients completely determine the solution.

# III. Radiative trapping

Thus far in our treatment of line radiation we have assumed that there is no background radiation field other than the CMB and starlight, which are imposed externally. In particular, we have assumed that photons emitted within a cloud all escape, and do not build up to produce their own radiation field. These are sometimes good assumptions, but there are environments where they fail, sometimes spectacularly. We have already encountered one, in our discussion of Lyman  $\alpha$  radiation in the context of recombination. In an ionised region where hydrogen is recombining, the occupation number of Lyman  $\alpha$  photons can be very large, and these photons are coming from the recombinations themselves, not from any external source. Treating problems of this type is highly non-trivial, and often we are forced to purely numerical solutions in which we iteratively solve simultaneously for the level populations and the radiation field. However, we can make some simple analytic approximations, which are sufficient to give at least rough accuracy in many circumstances.

# A. The escape probability formalism

Consider a gaseous region. At some point  $\mathbf{r}$  within it, the optical depth to infinity in direction  $\hat{\mathbf{n}}$  at frequency  $\nu$  is  $\tau_{\nu}(\hat{\mathbf{n}}, \mathbf{r})$ . We define the escape probability for photons of frequency  $\nu$  emitted at  $\mathbf{r}$  by

$$\beta_{\nu}(\mathbf{r}) = \frac{1}{4\pi} \int e^{-\tau_{\nu}(\hat{\mathbf{n}},\mathbf{r})} d\Omega$$
(33)

This is exactly what the name implies: it is the probability that a photon emitted in a random direction at point **r** will escape to infinity without being scattered or absorbed. If we integrate this over the line profile  $\phi_{\nu}$ , we obtain the probability that a random photon emitted in a particular line of interest will escape:

$$\langle \beta(\mathbf{r}) \rangle = \int \beta_{\nu}(\mathbf{r}) \phi_{\nu} \, d\nu. \tag{34}$$

Now consider a simple two-level system. For the upper state, the rate of change of the number density of particles is

$$\frac{dn_1}{dt} = \left( n_c k_{01} + \frac{g_1}{g_0} \langle n_\gamma \rangle A_{10} \right) n_0 - n_1 \left[ n_c k_{10} + \left( 1 + \langle n_\gamma \rangle \right) A_{10} \right], \tag{35}$$

where we have dropped the 10 subscript for  $n_{\gamma}$  for compactness, since there is only one frequency to worry about. The first term represents the rate at which the upper state is populated by collisions and absorptions out of the lower state, and the second represents the rate at which the upper state is depopulated due to collisions, spontaneous emission, and stimulated emission into the lower state.

In general solving this problem together with the radiative transfer problem is nasty, because  $n_{\gamma}$  varies throughout the cloud as dictated by the transfer equation. However, the source term in the transfer equation depends on the level populations, since they dictate the rate of emission and absorption. The level populations in turn depend on  $n_{\gamma}$ , so everything is non-locally coupled. This is why a numerical solution is usually the only option if we want to solve the problem "right".

However, if we want to solve the problem only sort of right, we can make some assumptions. Specifically, we will assume that: (1) the cloud is uniform, and (2) if a photon does not escape, it is re-absorbed very close to the point where it is emitted – an assumption known as the "on the spot" approximation. Assumption (1) means that the cloud can be characterised by a single excitation temperature  $T_{\text{ex}}$ , so the equation of radiative transfer through it becomes

$$I_{\nu} = I_{\nu}(0)e^{-\tau_{\nu}} + B_{\nu}(T_{\rm ex})(1 - e^{-\tau_{\nu}}).$$
(36)

Rewriting this in terms of the photon occupation number, this is

$$n_{\gamma} = n_{\gamma}(0)e^{-\tau_{\nu}} + \frac{1 - e^{-\tau_{\nu}}}{n_0 g_1/n_1 g_0 - 1},$$
(37)

where we have written  $T_{\text{ex}}$  in terms of the ratio of level populations  $n_1/n_0$ . Here  $n_{\gamma}(0) = (c^2/2h\nu^3)I_{\nu}(0)$  is the external radiation field entering the cloud from the outside.

Continuing with our assumption of uniformity, we can replace the  $e^{-\tau_{\nu}}$  factors with  $\beta_{\nu}$ , the escape probability that is, by assumption, the same everywhere. If we further assume that the external radiation field  $n_{\gamma}(0)$  is isotropic and does not vary significantly over the line profile, we can integrate the transfer equation over frequency and angle, giving

$$\langle n_{\gamma} \rangle = \langle \beta \rangle n_{\gamma}(0) + \frac{1 - \langle \beta \rangle}{g_1 n_0 / g_0 n_1 - 1}$$
(38)

If we now put this into our equation for the evolution of the level population, we have

$$\frac{dn_1}{dt} = n_c k_{01} n_0 - n_c k_{10} n_1 - \langle \beta \rangle A_{10} n_1 + n_0 \frac{g_1}{g_0} \langle \beta \rangle A_{10} n_\gamma(0) \left( 1 - \frac{g_0 n_1}{g_1 n_0} \right).$$
(39)

This is quite an interesting result, but it makes intuitive sense. The equation is exactly the same as what we got before for  $dn_1/dt$  assuming that every photon escaped, except that the Einstein coefficients have all been reduced by a factor of  $\langle \beta \rangle$ . This makes sense given our assumptions: since we have assumed that every photon has some probability of escaping, and that if it does not escape it is re-absorbed immediately at the point of emission, our assumptions essentially amount to reducing the rate of spontaneous emission by the escape probability. That is exactly what our analysis has produced. One could almost have guessed the result without going through the formalism at all.

Note that this means that in the case of radiative trapping we can also define an effective critical density, which is the same as the critical density in the optically thin case, but with all the A coefficients replaced by  $\langle \beta \rangle A$ .

## B. Approximate escape probabilities

Of course we have glossed over one critical point thus far: we have not yet specified exactly what the escape probability is! This is where things get tricky, since the exact escape probability will depend on the structure of the cloud in position and velocity. Any value will necessarily be approximate, since in a real cloud the escape probability is never uniform. A number of approximations exist, which are applicable to different velocity structures and geometries. For the purposes of this class, we will only consider one: spherical clouds with uniform velocity dispersions.

We already know how to compute the optical depth in this case from our discussion of absorption lines. Suppose we have a uniform, spherical cloud with velocity dispersion  $\sigma_v$  and radius R. In this case the optical depth from the centre to the edge is

$$\tau_0 = \frac{g_1}{g_0} \frac{A_{10} \lambda_{10}^3}{4(2\pi)^{3/2} \sigma_v} n_0 R \left( 1 - \frac{g_0 n_1}{g_1 n_0} \right), \tag{40}$$

where  $\lambda_{10} = hc/E_{10}$  is the wavelength of the transition. From any other point within the cloud, the optical depth is simply  $\tau_0$  multiplied by a geometric factor. In particular, if we consider a point a distance r from the cloud centre, and we consider a direction that makes an angle  $\theta$  with respect to the outward normal. Using a little geometry it is easy to see that the distance to the surface a is given implicitly by

$$R^2 = r^2 + a^2 + 2ar\cos\theta,\tag{41}$$

and the optical depth in this direction is  $x\tau_0$ , where x = a/R. The directionallyaveraged escape probability is therefore

$$\beta_{\nu}(r) = \frac{1}{2} \int_0^{\pi} e^{-x\tau_0\phi_{\nu}} \sin\theta \,d\theta.$$
(42)

Obviously this can be integrated in frequency as well in the usual way to obtain the frequency-averaged, angle-averaged escape probability  $\langle \beta \rangle$ . If we are not so optically thick that we need to worry about line wings, it is reasonable simply to take  $\phi_{\nu}$  to be a  $\delta$ -function, so

$$\langle \beta(r) \rangle = \frac{1}{2} \int_0^\pi e^{-x\tau_0} \sin\theta \, d\theta. \tag{43}$$

This still depends on position, but for the escape probability formalism we need to adopt a uniform value of  $\langle \beta \rangle$  throughout the cloud. The most natural way to do so is to simply to set this uniform value to the value of  $\langle \beta \rangle$  averaged over the cloud volume:

$$\langle\beta\rangle_{\rm cloud} = \frac{3}{2} \int_0^1 \int_0^\pi e^{-x\tau_0} x^2 \sin\theta \, d\theta \, dx. \tag{44}$$

For a given  $\tau_0$  this integral is trivial to evaluate numerically, but since we are working to an approximation anyway, it makes more sense to approximate it with something simple and analytic. It turns out that the numerical result is fit well by

$$\langle \beta \rangle_{\text{cloud}} \approx \frac{1}{1 + 0.5\tau_0}.$$
 (45)

The scalings here are easy to understand: for  $\tau_0 \to 0$ , most photons escape, and the escape probability is just unity. For  $\tau_0 \gg 1$ , photons are only likely to escape if they happen to be emitted within one optical depth of the cloud surface, so the escape probability is just the fraction of the cloud's volume within one optical depth of the surface. For large  $\tau_0$ , this fraction just scales as  $1/\tau_0$ , since for large  $\tau_0$  the volume from which photons can escape is just a thin skin of thickness  $R/\tau_0$  at the cloud surface. Our approximate expression just interpolates smoothly between these two limits.

We are now almost done: the value of  $\langle \beta \rangle_{\text{cloud}}$  that determines the equilibrium level populations is a known function of  $\tau_0$ , while  $\tau_0$  is a known function of the level populations. We therefore have a simple pair of non-linear algebraic equations to solve simultaneously, and we are done. In practice this solution must usually be obtained numerically, for example via Newton's method, but the problem is fairly straightforward.

C. Application of escape probability: the CO "X" factor

One very important application of the escape probability formalism is to emission by CO molecules. The reason is that galaxies keep a substantial part of their ISMs (as we shall see later in the course) in the form of molecular hydrogen. The H<sub>2</sub> molecule, however, is not directly observable under most circumstances, for reasons we will discuss. Therefore we tend to observe H<sub>2</sub> via a proxy, the most common of which are the rotational lines of the CO molecule, which tends to be found coincident with H<sub>2</sub>. The transitions we observe are changes in the rotational quantum number, for example  $J = 1 \rightarrow 0$  or  $J = 2 \rightarrow 1$ . It turns out that these transitions tend to be extremely optically thick, with small escape probabilities. Thus if we want to interpret CO emission, we need to think about escape probability.

Consider a uniform spherical cloud of mass M and radius R, containing CO molecules at kinetic temperature T. If the cloud is in hydrostatic balance between internal pressure (mostly due to turbulent rather than thermal motions, but that doesn?t matter for our purposes) and self-gravity, application of the virial theorem

shows that the 1d velocity dispersion is  $\sigma_v \approx \sqrt{GM/5R}$ . What emission do we expect to see?

As a first step, we can compute the optical depth from cloud centre to edge:

$$\tau_0 = \frac{g_1}{g_0} \frac{A_{10} \lambda_{10}^3}{4(2\pi)^{3/2} \sigma_v} n_0 R \left( 1 - \frac{g_0 n_1}{g_1 n_0} \right) = \frac{g_1}{g_0} \frac{A_{10} \lambda_{10}^3}{8\pi} \left( \frac{5}{2\pi G} \right)^{1/2} n_0 \frac{R^{3/2}}{M^{1/2}} \left( 1 - \frac{g_0 n_1}{g_1 n_0} \right)$$
(46)

where here we take states 0 and 1 to the be the J = 1 and J = 0 states (though we could equally well choose J = 2 and J = 1), so  $n_0$  is the number density of CO molecules in the J = 0 state. Using the escape probability approximation, whereby we just replace  $A_{10}$  with  $\langle \beta \rangle_{\text{cloud}} A_{10}$  and assume the cloud to be uniform, the intensity that we observe along a line of sight whose path length through the cloud is L is

$$I_{\nu} = n_1 \langle \beta \rangle_{\text{cloud}} A_{10} \frac{h\nu_{10}}{4\pi} \phi_{\nu} L.$$
(47)

Plugging in our expression for the approximate escape probability, this is

$$I_{\nu} = n_1 A_{10} \frac{h\nu_{10}}{4\pi} \phi_{\nu} L \frac{1}{1+0.5\tau_0} \approx \frac{2}{\tau_0} n_1 A_{10} \frac{h\nu_{10}}{4\pi} \phi_{\nu} L, \qquad (48)$$

where the second step applies if  $\tau_0 \gg 1$ , which is the case here. Now notice that  $\tau_0 \propto n_0$ , so  $I_{\nu}$  does not end up depending on  $n_1$  or  $n_0$  separately, only on  $n_1/n_0$ . This makes physical sense: if the cloud is very optically thick, then when we look at it we should see an intensity given by the Planck function, independent of the total amount of material we are looking at. If we are in LTE, which is a good assumption most of the time, then we can rewrite  $n_1/n_0$  just in terms of T.

Since this is a radio observation, it is more common to work in terms of antenna temperature or brightness temperature (which are effectively the same thing, since  $h\nu \ll kT$  here):  $T_A = (\lambda_{10}^3/2k)I_{\nu}$ . Radio astronomers also insist on working in velocity instead of frequency, where  $v = c(\lambda/\lambda_{10})$ , i.e., v is just the velocity that corresponds to a given shift in wavelength or frequency. Thus the most common way that observations of the CO line are reported is in terms of the velocity-integrated antenna temperature,

$$\int T_A \, dv = \sqrt{\frac{2}{15}} 4\pi \frac{hc}{\lambda} \left(\frac{G\mu m_{\rm H}}{n_{\rm H_2}}\right)^{1/2} \frac{N_{\rm H_2}}{e^{h\nu/kT} - 1},\tag{49}$$

where we have rewritten the total cloud mass as  $M = (4/3)\pi R^3 \mu m_{\rm H} n_{\rm H_2}$ , the column density  $N_{\rm H_2} = M/(\pi R^2 \mu m_{\rm H})$ ,  $n_{\rm H_2}$  is the number density of H<sub>2</sub> molecules, T, and  $\mu = 2.8$  is the mean mass per H<sub>2</sub> molecule in units of hydrogen masses  $m_{\rm H}$ (2.8 rather than 2 because of helium). Note that this means that the velocityintegrated antenna temperature is just proportional to the column density of H<sub>2</sub>. We invert this relationship to define the constant of proportionality  $X_{\rm CO}$ :

$$X_{\rm CO} = \frac{N_{\rm H_2}}{\int T_A \, dv} = 1.6 \times 10^{20} \left(\frac{n_{\rm H_2}}{10^3 \, \rm cm^{-3}}\right)^{1/2} \left(e^{5.5 \,\rm K/T} - 1\right) \,\,\rm cm^{-2}/\left(\rm K \,\,\rm km \,\,s^{-1}\right),\tag{50}$$

where the numerical evaluation is for the  $J = 1 \rightarrow 0$  line; other lines can be treated identically, just by substituting in the correct wavelength and value of  $h\nu$ . What this means is that, to the extent that  $n_{\rm H_2}$  and T do not vary wildly between molecular clouds, this provides a simple way of turning an observed CO luminosity into an estimate of the corresponding column density of H<sub>2</sub>.