Thus far we have considered only nuclear burning that happens in the cores of stars. However, as we progress to post-main sequence evolution, we will often encounter a different configuration: stars that have burned all of the material in their cores to inert ash, but still have fuel left in their outer layers. In this configuration, the star will burn fuel in a thin shell around the inert core. However, as we will show in this tutorial, such a configuration tends to be unstable.

We will consider a thin shell located at a distance $r_{\rm sh}$ from the centre of the star. The shell has a thickness dr and a mass dm. There is a mass $m_{\rm sh}$ interior to the shell. The density and temperature of the gas in the shell are ρ and T, respectively. The star as a whole has radius R and mass M. From hydrostatic balance, $dP/dm = -Gm/4\pi r^4$, the pressure in the shell must be

$$P_{\rm sh} = -\int_{m_{\rm sh}}^{M} \frac{Gm}{4\pi r^4} dm.$$

Now consider what happens if we perturb the outer radius of the shell outward by a distance $\delta r \ll dr$ (i.e., by an amount much less than the shell thickness), while leaving the inner radius unchanged. Every shell of material outside the shell must move outward proportionately, so material that was previously at radius r moves to radius $r' = r(1 + \delta r/r_{\rm sh})$.

Exercise 1. *Use hydrostatic balance to derive the amount δP by which the pressure in the shell is altered by this perturbation. You should be able to show that*

$$\frac{\delta P}{P_{\rm sh}} = k_1 \frac{\delta r}{r_{\rm sh}}$$

for some constant k_1. Find the value of k. Hint: make use of the fact that $\delta r \ll r_{\rm sh}$ to Taylor expand terms involving $\delta r/r_{\rm sh}$.

Next we will consider how the density in the shell responds to this perturbation. The new shell thickness is $dr + \delta r$, but the mass in the shell remains unchanged. The density, mass, and radius are related by the usual relationship $dr/dm = 1/4\pi r^2 \rho$.

Exercise 2. *Use the relationship between mass, radius, and density to derive the amount $\delta\rho$ by which the density of the shell changes in response to the thickness changing from dr to $dr + \delta r$. As with pressure, you should be able to find a relationship*

$$\frac{\delta \rho}{\rho} = k_2 \frac{\delta r}{dr}$$

for some k_2. Find the value of k_2. As in exercise 1, make use of the fact that $\delta r/dr \ll 1$.

The next step is to combine the two results above to find a relationship between the perturbation in pressure and the perturbation in density.

Exercise 3. *Combine your results for exercises 1 and 2 to show that the pressure and density perturbations are related by*

$$\frac{\delta P}{P_{\rm sh}} = 4 \frac{dr}{r_{\rm sh}} \frac{\delta \rho}{\rho}.$$

Since the rate of nuclear burning depends on the temperature, the next step is to understand how this influences the temperature. The temperature will change from its original value T to a perturbed value $T + \delta T$. Suppose the gas obeys an equation of state whereby $P \propto \rho^a T^b$. For example, an ideal gas has $a = 1$, $b = 1$, while a non-relativistic degenerate gas has $a = 4/3$, $b \approx 0$ (not exactly 0, since $b = 0$ exactly is achieved only at zero temperature).

Exercise 4. *Use the equation of state to show that the density, pressure, and temperature perturbations are related by*

$$\frac{\delta P}{P} = a\frac{\delta\rho}{\rho} + b\frac{\delta T}{T}.$$

Now we're ready to put everything together.

Exercise 5. *Combine your expressions for $\delta P/P$ and $\delta P/P_{\rm sh}$ from exercises 3 and 4 to show that*

$$\left(4\frac{dr}{r_{\rm sh}} - a\right)\frac{\delta\rho}{\rho} = b\frac{\delta T}{T}.$$

The coefficient $4(dr/r_{\rm sh} - a)$ is always negative for a thin shell, since $dr/r_{\rm sh} \ll 1$, while a and b are always positive (pressure increases with both density and temperature). Thus if the density goes down by some fractional amount ($\delta\rho/\rho < 0$), the temperature will rise by a similar fractional amount to compensate ($\delta T/T > 0$).

Exercise 6. *Recall that most nuclear reactions are very, very sensitive to temperature, much more so than they are to density. Use this combined with the above relationship to argue that thin shells should be unstable. As a hint, consider what happens if a shell burns a little bit too fast to be in equilibrium, causing it to expand slightly. The expansion will make its density drop. What happens to the rate of nuclear burning as a result? Does it go up or down? Does that make the burning stable or unstable?*