

As we discussed at the end of last class, we've written down the evolution equations, but we still need to specify how to fill in the things like pressures, reaction rates, rates of energy transfer through the star, etc. Today we're going to tackle the problem of pressure in stars, also known as the equation of state: an equation that specifies the pressure in a gas given its density and temperature. You're all familiar with the most common of these, the ideal gas law: $P = nk_B T$. While this works well under terrestrial conditions, inside a star things get a bit trickier. To derive the equation of state for a star, we will need to talk a little about the kinetic theory of gasses.

I. Pressure in the kinetic theory of fluids

In order to compute the pressure of stellar material, we need to recall that pressure is the force exerted by a gas on a surface, such as the walls of its container, and that force is a momentum per unit time. In other words, the pressure is the momentum per unit time per unit area that a gas transfers to the walls of the vessel containing it. The reason there is a momentum transfer is that particles in the gas are moving around at random, and that some of them will strike the walls of the vessel, bounce off, and transfer momentum. We can compute the pressure by computing this momentum transfer.

To see what this implies, consider an immobile surface with a gas on one side of it, and focus on an area of that surface dA . First consider a single particle with momentum p approaching the surface at an angle θ relative to the normal and bouncing off it elastically (Figure 1, left). A little geometry quickly shows that the momentum transferred to the surface is $2p \cos \theta$.

Now consider a beam of particles, all moving toward the surface at angle θ and bouncing off, and all moving with the same momentum p (Figure 1, right). Suppose the number density of particles in the beam is n , and that they are moving at velocity v (which is related to p in the usual way). The rate at which particles strike the surface is $nv \cos \theta dA$. The $nv dA$ comes from multiplying the density of particles available by the speed at which they move by the area available to catch them. You can understand the factor of $\cos \theta$ in two equivalent ways. One is that only a fraction $\cos \theta$ of that velocity is in the direction perpendicular to the surface, and velocity parallel to the surface doesn't produce any collisions. Alternately, you can think about the projected area of the surface as seen by a particle in the beam, which is smaller than its total area by a factor $\cos \theta$. Since each collision transfers a momentum $2p \cos \theta$, the total rate at which the beam transfer momentum to the surface is

$$\frac{d^2 p_{\text{surf}}}{dt dA} = 2nvp \cos^2 \theta. \quad (1)$$

To generalise from the case of a beam to the case of a gas, we have to consider the fact that particles are moving in every possible direction. Continuing for the moment to imagine that all particles have the same momentum, the n be the total number density of particles, and let $dn(\theta)/d\theta$ be the number density of particles coming in at angles between θ and $\theta + d\theta$ relative to the normal. If the particle distribution is isotropic, then the fraction of particles at angle θ is just proportional to the fraction of the solid angle that lies between θ and $\theta + d\theta$. Figure 2 shows the geometry. The solid angle of the indicated strip in the figure is $2\pi \sin \theta d\theta$, as compared to 4π sr in total, so we must have that

$$\frac{dn(\theta)}{d\theta} = \frac{1}{2}n \sin \theta. \quad (2)$$

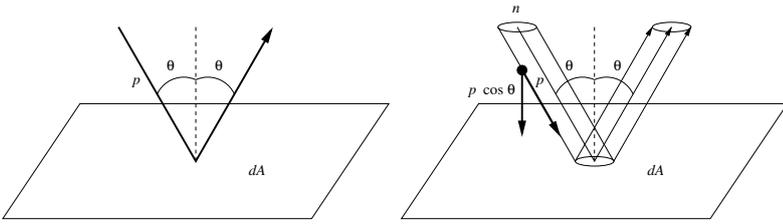


Figure 1: Geometry of particles bouncing off a surface. The left panel shows a single particle with momentum p reflecting off the surface, while the right panel shows a beam of particles of number density n .

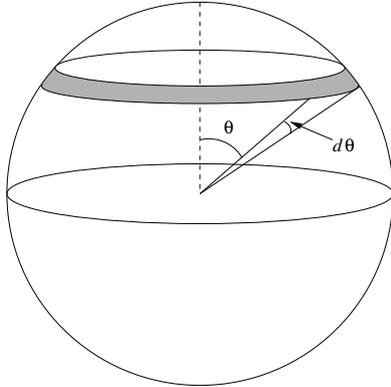


Figure 2: Geometry of particles approaching a surface from a particular solid angle.

Thus the collision rate for particles coming in at angle θ is $(dn(\theta)/d\theta)v \cos \theta dA$, and each collision still transfers momentum $2p \cos \theta$. To get the total rate of momentum transfer we just have to multiply collision rate times momentum transfer and integrate this over all angles:

$$\frac{d^2 p_{\text{surf}}}{dt dA} = npv \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta \quad (3)$$

$$= npv \int_0^1 \cos^2 \theta d \cos \theta \quad (4)$$

$$= \frac{1}{3} npv \quad (5)$$

Finally, to generalise this to a distribution of particles that aren't all moving at the same speed, we just have to integrate over their momentum distribution. We let $dn(p)/dp$ be the number of particles with momenta between p and $p + dp$. The pressure is then simply the momentum transferred to the surface per unit time per unit area, which we obtain simply by integrating over all the possible particle momenta:

$$P \equiv \frac{d^2 p_{\text{surf}}}{dt dA} = \frac{1}{3} \int_0^{\infty} \frac{dn(p)}{dp} pv dp \quad (6)$$

II. Types of Pressure

We have now written the pressure of a gas in terms of the momentum distribution of its particles. This required a lengthy mathematical exercise, but this was worth it because this enables us to define the pressure in all sorts of complicated situations where we can't blindly apply the ideal gas law. There are several relevant for stars that we'll consider next.

A. Re-derivation of the ideal gas law

The first step in our analysis is to use this kinetic model of pressure to re-derive the ideal gas law. This will provide a guide to how to proceed when dealing with

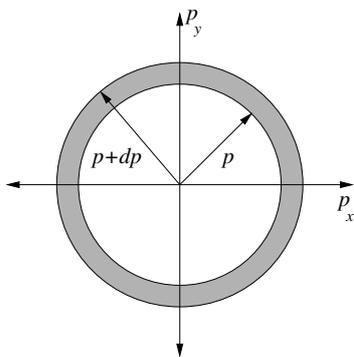


Figure 3: A two-dimensional momentum space.

more complicated situations. To derive the ideal gas law, we begin with a gas whose particles all have mass m , and where the particles have a Maxwell-Boltzmann velocity distribution, whereby the probability that a particle is in a state with energy E is proportional to $e^{-E/k_B T}$.

To derive the momentum distribution from this, it is helpful to first think about things two-dimensionally. A gas particle can have any vector momentum, \mathbf{p} . In two dimensions, this has two components, p_x and p_y . We can think of the momentum of a given particle as corresponding to a point in the two-dimensional plane of p_x and p_y (Figure 3).

We want to know the probability that a particle will be at a point (p_x, p_y) in this plane, and the Boltzmann distribution can tell us. The energy and momentum of a particle are related by

$$E = \frac{p^2}{2m} = \frac{p_x^2 + p_y^2}{2m}, \quad (7)$$

where p is the magnitude of the vector \mathbf{p} . Thus the probability of being at a point (p_x, p_y) is proportional to $e^{-(p_x^2 + p_y^2)/(2mk_B T)}$. The probability that the magnitude of the momentum will fall in the ring between p and $p + dp$ is just the probability of being at a point (p_x, p_y) in the ring times the area of the ring, which is $2\pi p dp$. Thus in two dimensions we have

$$\frac{dn(p)}{dp} \propto 2\pi p e^{-p^2/(2mk_B T)}. \quad (8)$$

The three-dimensional generalization is obvious: instead of a ring of area $2\pi p dp$, we now have a shell of volume $4\pi p^2 dp$. Thus in three dimensions the momentum distribution for the particles must follow

$$\frac{dn(p)}{dp} \propto 4\pi p^2 e^{-p^2/(2mk_B T)}. \quad (9)$$

To get the normalisation constant, we just require that, when we integrate over all momenta, we get the right number of particles. Thus we say that $dn(p)/dp = k \cdot 4\pi p^2 e^{-p^2/(2mk_B T)}$ and solve for the constant k by requiring that

$$n = 4\pi k \int_0^\infty p^2 e^{-p^2/(2mk_B T)} dp \quad (10)$$

$$= 4\pi k (2mk_B T)^{3/2} \int_0^\infty x^2 e^{-x^2} dx \quad (11)$$

$$= \pi^{1/2} k (2mk_B T)^{3/2} \quad (12)$$

where in the second step we have made the substitution $x = p/\sqrt{2mk_B T}$, and in the third step we evaluated the integral to get $\sqrt{\pi}/4$ – the integral is fairly

straightforward to do by standard tricks. This gives us k , which in turn gives us $dn(p)/dp$:

$$\frac{dn(p)}{dp} = \frac{4}{\sqrt{\pi}} \frac{n}{(2mk_B T)^{3/2}} p^2 e^{-p^2/(2mk_B T)} \quad (13)$$

Given this result, computing the pressure is just a matter of plugging into [Equation 6](#) and evaluating the integral:

$$P = \int_0^\infty \frac{1}{3} \left[\frac{4}{\sqrt{\pi}} \frac{n}{(2mk_B T)^{3/2}} p^2 e^{-p^2/(2mk_B T)} \right] p \left(\frac{p}{m} \right) dp \quad (14)$$

$$= \frac{4n}{3\pi^{1/2}m} (2mk_B T) \int_0^\infty x^4 e^{-x^2} dx \quad (15)$$

$$= nk_B T, \quad (16)$$

where in the last step we again evaluated the integral, this time to $3\sqrt{\pi}/8$. Thus we have successfully re-derived the ideal gas law from first principles using the kinetic theory of gasses.

B. Multicomponent ideal gasses

The first complication to add to this story is what happens if we have multiple types of particles, each with a different mass. This is relevant to a gas that contains a mixture of hydrogen and helium, for example. It is also relevant in a fully ionised gas, where the ions and electrons move separately, and obviously their masses are quite different. Fortunately, the kinetic description makes the result obvious: each species follows the Boltzmann distribution, and the sum of the momentum transferred to a surface is simply the sum of the momenta transferred by the particles of each species, each of which is given by $nk_B T$. Thus, if we have N species present in the gas, then the total pressure is simply

$$P = \left(\sum_{i=1}^N n_i \right) k_B T. \quad (17)$$

We can write this equivalently in terms of the mass fraction and mass. If we let $\mathcal{A}_i m_H$ be the mass per particle of species i and X_i be the fraction of the mass at a given point that comes from species i , then, as before, we have

$$n_i = \frac{X_i}{\mathcal{A}_i m_H} \rho, \quad (18)$$

and therefore we can write the pressure as

$$P = \left(\sum_{i=1}^N \frac{X_i}{\mathcal{A}_i} \right) \rho \mathcal{R} T \quad (19)$$

For convenience we define

$$\frac{1}{\mu} = \sum_{i=1}^N \frac{X_i}{\mathcal{A}_i}, \quad (20)$$

where μ is the mean mass (measured in units of hydrogen masses) per particle, so that the ideal gas law becomes

$$P = \frac{\mathcal{R}}{\mu} \rho T. \quad (21)$$

If we only include ions (not electrons) in the sum, then we get the pressure due to ions alone, and we write μ in this case as μ_I , for the mean mass per particle of

ions. Since the Sun is mostly hydrogen and helium, it is convenient to express its composition in terms of the fraction of the mass that is hydrogen, the fraction that is helium, and the fraction that is everything else – the everything else we call metals. Note that, to an astronomer, carbon, oxygen, and neon are all metals! We define X as the hydrogen mass fraction, Y as the helium mass fraction, and Z as the metal mass fraction. For the Sun, $X = 0.707$, $Y = 0.274$, and $Z = 1 - X - Y = 0.019$.

We can write μ_I in terms of these definitions:

$$\frac{1}{\mu_I} = \frac{X}{1} + \frac{Y}{4} + \frac{Z}{\langle \mathcal{A} \rangle_{\text{metals}}}, \quad (22)$$

where $\langle \mathcal{A} \rangle_{\text{metals}}$ is the mean atomic mass of the metals, which is about 20 in the Sun. Thus for the Sun $\mu_I = 1.29$.

We can similarly calculate the pressure due to electrons. In the outer layers of a star where it is cold there are no free electrons, but we showed using the virial theorem that in the stellar interior the gas is close to fully ionised. Thus there is one free electron per proton. If n_i is the number density of ions of species i , then the number density of electrons is

$$n_e = \sum_i Z_i n_i = \frac{\rho}{m_H} \sum_i X_i \frac{Z_i}{\mathcal{A}_i}. \quad (23)$$

Again, for convenience we give this sum a name:

$$\frac{1}{\mu_e} = \sum_i X_i \frac{Z_i}{\mathcal{A}_i}. \quad (24)$$

The meaning of μ_e is that it is the average number of free electrons per nucleon, meaning per proton or neutron. In terms of our X , Y , and Z numbers,

$$\frac{1}{\mu_e} = X + \frac{Y}{2} + Z \left\langle \frac{Z}{\mathcal{A}} \right\rangle_{\text{metals}}, \quad (25)$$

where the term $\langle Z/\mathcal{A} \rangle_{\text{metals}}$ represents the ratio of electrons (or protons) averaged over all the metal atoms. This is roughly 1/2, so to good approximation

$$\frac{1}{\mu_e} \simeq X + \frac{Y}{2} + \frac{Z}{2} = \frac{1}{2}(X + 1), \quad (26)$$

since $Z = 1 - X - Y$. Thus for the Sun $\mu_e = 1.17$.

Thus the pressures of the ions and the electrons are $P_I = (\mathcal{R}/\mu_I)\rho T$ and $P_e = (\mathcal{R}/\mu_e)\rho T$, so the total pressure is $P = P_I + P_e = (\mathcal{R}/\mu)\rho T$, where

$$\frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}. \quad (27)$$

C. Relativistic gasses and radiation

The rule that pressures from different gasses just add is fairly intuitive, and one could probably have guessed it without the kinetic theory. We do need the kinetic theory, however, to generalize the concept of pressure to gasses that are not ideal, classical gasses. The simplest generalization to make is to gases that are relativistic, meaning that the particles within them are moving at close to the speed of light. This occurs in some extreme stars. We will limit ourselves to considering gases in the

extremely relativistic limit, where most particles have speeds very close to c . The partially relativistic case is conceptually the same, but involves a great deal more algebra.

For a relativistic gas the pressure is defined via [Equation 6](#) exactly the same as for a non-relativistic one. There are only two differences. The first is that the velocity v in the integral is nearly c , the speed of light. The second is that energy and momentum are no longer related by $E = p^2/(2m)$, and thus the momentum distribution $dn(p)/dp$ corresponding to the Boltzmann distribution is different. For particles of mass m , this relationship between energy and momentum begins to change when the mean particle energy, $(3/2)k_B T$, becomes comparable to $(1/2)mc^2$, the particle rest energy. In the extreme relativistic limit, where the particle rest energy is small compared to its kinetic energy, the energy and momentum of a particle are related by

$$E = pc. \quad (28)$$

To compute the pressure, we need only compute the momentum distribution that this relationship between E and p requires. This is

$$\frac{dn(p)}{dp} = 4\pi k p^2 e^{-E/k_B T} = 4\pi k p^2 e^{-pc/k_B T}, \quad (29)$$

and the constant k is again determined by requiring that

$$n = 4\pi k \int_0^\infty p^2 e^{-pc/k_B T} dp = 8\pi k \left(\frac{k_B T}{c} \right)^3 \quad (30)$$

$$k = \left(\frac{c}{k_B T} \right)^3 \frac{n}{8\pi} \quad (31)$$

The pressure is

$$P = \frac{n}{6} \frac{c^4}{(k_B T)^3} \int_0^\infty p^3 e^{-pc/k_B T} dp \quad (32)$$

$$= nk_B T \quad (33)$$

This is exactly the same as for a non-relativistic gas.

However, for relativistic gasses we have a complication which is not present for non-relativistic ones, which is that the number of particles is not necessarily fixed. Instead, when particles are moving around with an energy comparable their rest energy, collisions can create or destroy particles. Thus n and $dn(p)/dp$ are no longer fixed, and instead becomes functions of T . Our result is valid only for fixed n .

We will not solve this problem in general, but we will solve it for one particular type of relativistic gas: radiation. We can think of photons as a relativistic gas, since photons move at the speed of light and have energies much larger than their rest energy (which is zero). To compute the pressure of a photon gas, we need to know how the number density of photons and its distribution in momentum, $dn(p)/dp$, varies with the temperature T . This distribution is

$$\frac{dn(\nu)}{d\nu} = \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} \quad (34)$$

$$\frac{dn(p)}{dp} = \frac{8\pi p^2}{h^3} \frac{1}{e^{pc/k_B T} - 1}. \quad (35)$$

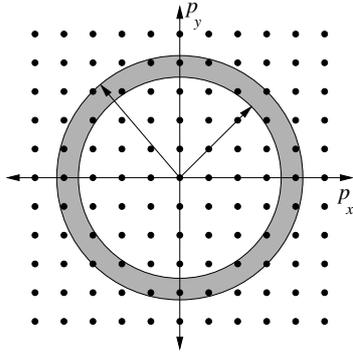


Figure 4: Quantisation of momentum space.

The first form is in terms of the frequency, and the second is in terms of the momentum. The two are related by $E = pc = h\nu$. This distribution, known as Planck's Law, was first found empirically by Max Planck in 1901 and was finally understood theoretically by Satyendra Nath Bose in 1924. An interesting historical aside: Bose was a professor at the University of Dhaka in India, and when he first produced this result, no journal in Europe was willing to accept his paper. Eventually he sent the paper to Einstein, who recognized its significance and wrote a companion paper in support of Bose's. The two were then published together, giving rise to what is known today as Bose-Einstein statistics.

Given this distribution, we derive the pressure as before:

$$P = \frac{1}{3} \int_0^\infty c \frac{h\nu}{c} \frac{dn(\nu)}{d\nu} dp = \frac{1}{3} aT^4, \quad (36)$$

where

$$a = \frac{8\pi^5 k_B^4}{15c^3 h^3} = \frac{4\sigma}{c} \quad (37)$$

is known as the radiation constant.

D. Degenerate gasses

The second generalisation we will consider is to gasses where quantum mechanical effects become important. These are called degenerate gasses. A full theory of degenerate gasses and their pressures is beyond what we will do in this class, but we will deal with one limiting case, and we can use that to provide a good approximation to the quantum effect.

Consider again the picture of the (p_x, p_y) plane, where we describe every particle's momentum in terms of a position in the plane. In classical mechanics, a particle can occupy any position in the (p_x, p_y) plane, but quantum mechanics tells us that in reality there actually only discrete, quantized values of p_x and p_y that particles are allowed to have – in effect there is a grid in (p_x, p_y) -space, and particles can only be found on the grid points (Figure 4).

Most of the time this doesn't matter, because the grid points are packed so densely that they might as well be a continuum. Particles can't really be anywhere, but they can be nearly anywhere. However, there are some situations where it does matter. In the classical picture, the probability of being at a given point is $e^{-E/k_B T}$, where E is the energy associated with that point. This distribution continues to apply in quantum mechanics. If T is small, then $E/k_B T$ is a big number for most grid points, so the particles all try to crowd into the points close to the origin, where E is small. As a result, they're all trying to occupy the same few grid points. However, the Pauli

exclusion principle says that no two fermions (a category of particles that includes electrons) can occupy the same quantum state. For electrons, which can be spin up or spin down, no more than two can sit at any grid point. Because the electrons can't all pack into the few central grid points, they are forced to occupy a wider range of momenta than classical mechanics would suggest they should. As a result, their pressure is much higher than you would expect based on classical mechanics.

To know when this effect is important, we need to know what the density of grid points is, since that will dictate when we start to have problems with too many electrons trying to sit at the same site. Actually calculating this rigorously is beyond the scope of this class, but one can obtain the basic result from the Heisenberg uncertainty principle. The most common way of stating this is that there is an irreducible uncertainty in the product of a particle's momentum and its position:

$$\Delta x \Delta p \geq h, \quad (38)$$

where $h = 6.63 \times 10^{-27}$ erg s is Planck's constant. In 3D, we can write this as

$$\Delta V \Delta^3 p \geq h^3. \quad (39)$$

This relates the uncertainty in the volume where a particle is located to the uncertainty in its 3D momentum. An equally valid interpretation of the Heisenberg uncertainty principle is that it tells us how tightly packed the quantum grid points are. If we have a volume of space ΔV , then the grid points for particles in that volume each occupy a space $\Delta^3 p = h^3 / \Delta V$ in the (p_x, p_y, p_z) -space.

This tells us that quantum effects are going to start become important in two circumstances. One is when the temperature is low, and all the particles try to pack into the inner few gridpoints. The other is when the density of particles is high. This is because a high density means a large number of particles in a small space ΔV . However, when ΔV is small, then the quantum grid points are spaced a larger distance apart, which means there are few sites available for particles to occupy.

To apply this idea to calculating the pressure of a gas, consider the limit of a gas where the temperature approaches 0. In this case, the particles will try to crowd as close to the origin in (p_x, p_y, p_z) -space as possible. One can imagine placing the particles at the grid points. The first two electrons will go at the grid point closest to the origin, the next two and the second closest point, and so forth until all the electrons are used up. Thus the particles fill a circle of radius p_F in the (p_x, p_y) -plane in the 2D case, or a sphere in the (p_x, p_y, p_z) -volume in the 3D case. All the grid points with momentum $p < p_F$ are occupied, and all the grid points further from the origin than p_F will be empty. A gas of this sort is fully degenerate, meaning that the particles are packed as closely as possible. The quantity p_F is known as the Fermi momentum.

To get the pressure in this fully degenerate state, we need to know the momentum distribution $dn(p)/dp$ – that is, we need to know how many electrons there are inside the shell from p to $p + dp$. For the fully degenerate case this is easy. If $p > p_F$, then $dn(p)/dp = 0$, because all the grid points at $p > p_F$ are empty. If $p < p_F$, then all the grid points are full, so the number of electrons is just twice the number of grid points within the shell (since there are two electrons per grid point). Since the shell has volume $4\pi p^2 dp$, and each grid point takes up a volume $\Delta^3 p = h^3 / \Delta V$, the number of electrons inside the shell is

$$N_e = 2 \frac{4\pi p^2 dp}{\Delta^3 p} = \frac{2}{h^3} 4\pi p^2 dp \Delta V. \quad (40)$$

To change this to a number density, we just divide both sides by ΔV , which gives

$$\frac{dn(p)}{dp} = \frac{2}{h^3} 4\pi p^2 \quad (41)$$

To figure out the Fermi momentum p_F , we simply set it by the condition that, when we integrate over all momenta, we get the right number of particles:

$$n = \int_0^{p_F} \frac{2}{h^3} 4\pi p^2 dp \quad (42)$$

$$= \frac{8\pi}{3h^3} p_F^3 \quad (43)$$

$$p_F = \left(\frac{3h^3 n}{8\pi} \right)^{1/3}. \quad (44)$$

Finally, we are in a position to calculate the pressure. Suppose that all the particles have mass m . Then

$$P = \frac{1}{3} \int_0^\infty \frac{dn(p)}{dp} p v dp \quad (45)$$

$$= \frac{1}{3} \int_0^{p_F} \left(\frac{2}{h^3} 4\pi p^2 \right) p \left(\frac{p}{m} \right) dp \quad (46)$$

$$= \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{20m} n^{5/3} \quad (47)$$

$$= \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{20m_e m_H^{5/3}} \left(\frac{\rho}{\mu_e} \right)^{5/3}, \quad (48)$$

where in the last step we have assumed that the particles are electrons, and we have inserted the electron density for a fully ionized gas. The result applies equally well to protons and neutrons, since they are fermions too, but since the degeneracy pressure varies as $1/m$, the much higher mass of these particles means that their degeneracy pressure is much lower. Thus we are generally concerned only with electrons. The combination of constants in front of the ρ term comes up often enough that it is useful to compute it. We define

$$K'_1 = \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{20m_e m_H^{5/3}} = 1.00 \times 10^{13} \text{ dyn cm}^{-2} (\text{g cm}^{-3})^{-5/3}, \quad (49)$$

so that $P = K'_1 (\rho/\mu_e)^{5/3}$.

This is the pressure of a fully degenerate gas, and it represents a lower limit on the pressure, which is achieved at zero temperature. In reality at any finite temperature the pressure is higher than this, and those of you enrolled in ASTR4007/6007 will compute the result for finite temperature on your homework. As a very crude approximation, we can write the electron pressure as

$$P_e = \max \left[\mathcal{R} \frac{\rho}{\mu_e} T, K'_1 \left(\frac{\rho}{\mu_e} \right)^{5/3} \right], \quad (50)$$

i.e., the electron pressure is either the thermal pressure or the degeneracy pressure, whichever is greater. The transition between degenerate and non-degenerate occurs roughly where these two pressures are equal, which requires that

$$\frac{\rho}{\mu_e} = \left(\frac{\mathcal{R}}{K'_1} T \right)^{3/2} = 750 \left(\frac{T}{10^7 \text{ K}} \right)^{3/2} \text{ g cm}^{-3}. \quad (51)$$

One subtle but important thing to notice is that the degeneracy pressure, unlike the thermal pressure, does not depend on the gas temperature – a degenerate gas has essentially fixed pressure until the temperature rises high enough to make it non-degenerate. We will see that this has profound consequences for the evolution of degenerate stars. It causes some of them to explode.

E. Relativistic degenerate gasses

In some very dense stars, the gas is degenerate, and it is also dense enough so that the electrons have speeds that approach the speed of light. In this case we have a relativistic degenerate gas. Again, the procedure to calculate the pressure is the same, except that the velocity is now c , and the energy and momentum are related by $E = pc$. In the degenerate case, however, the change in the relationship between energy and momentum doesn't matter, because the momentum distribution is dictated by how many particles you can pack into a given volume in momentum-space, not by Boltzmann factors. Thus $dn(p)/dp$ is the same as for the non-relativistic case, and we have

$$P = \frac{1}{3} \int_0^{p_F} \left(\frac{2}{h^3} 4\pi p^2 \right) pc dp = \frac{2\pi c}{3h^3} p_0^4 = \left(\frac{3}{\pi} \right)^{1/3} \frac{hc}{8} n^{4/3} = \left(\frac{3}{\pi} \right)^{1/3} \frac{hc}{8m_{\text{H}}^{4/3}} \left(\frac{\rho}{\mu_e} \right)^{4/3}. \quad (52)$$

Again, in the last step we have assumed that the particles in question are electrons.

As with the non-relativistic degenerate case, it is convenient to give the collection of constants a name, so we say that

$$P = K'_2 (\rho/\mu_e)^{4/3}, \quad (53)$$

where $K'_2 = 1.24 \times 10^{15} \text{ dyn cm}^{-2} (\text{g cm}^{-3})^{-5/3}$. Again, note that the pressure does not depend on temperature.

The condition for a degenerate gas to be relativistic is that p_0 must be large enough so that the kinetic energy is comparable to the rest energy of the electron. Thus the gas becomes relativistic when $p_F^2/(2m_e) \sim m_e c^2$. This requires that

$$\frac{\rho}{\mu_e} = \frac{16\pi\sqrt{2} m_{\text{H}} m_e^3 c^3}{3 h^3} = 3 \times 10^6 \text{ g cm}^{-3}. \quad (54)$$

The condition for a relativistic gas to become degenerate is that the degeneracy pressure equal the gas pressure, which requires that

$$\frac{\rho}{\mu_e} \mathcal{R}T = K'_2 \left(\frac{\rho}{\mu_e} \right)^{4/3} \quad (55)$$

$$\frac{\rho}{\mu_e} = \left(\frac{\mathcal{R}T}{K'_2} \right)^3 = 0.3 \left(\frac{T}{10^7 \text{ K}} \right)^3 \text{ g cm}^{-3} \quad (56)$$

F. Regimes of pressure

We summarise the four cases we have just derived for relativistic and non-relativistic, degenerate and non-degenerate gasses in [Figure 5](#), by combining the conditions for switching between the regimes. The plot indicates where each case applies.

One thing that initially seems surprising about this plot is that it seems to suggest that fluid at a density of 1 g cm^{-3} should be degenerate unless its temperature is more

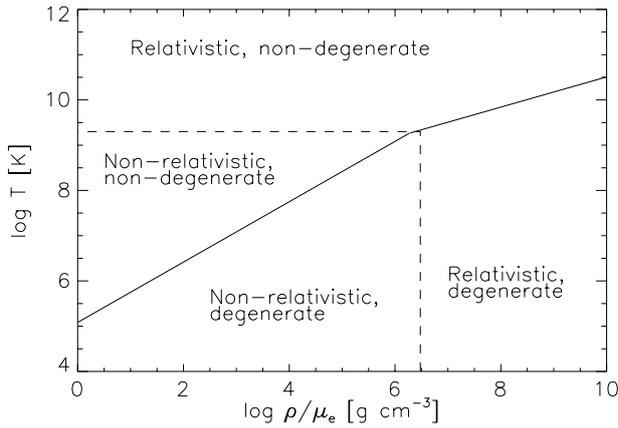


Figure 5: Regimes of pressure.

than 10^5 K or so. Does this mean that water at room temperature is degenerate? The answer is no. Recall that this plot is for electrons. A gas of free electrons with the density of water and a temperature comparable to room temperature would indeed be degenerate. However, water molecules are not free electrons. The H_2O molecule has a mass of $18m_{\text{H}}$, which is 3.3×10^4 electron masses. Recall that degeneracy pressure varies as $1/m$, so the degeneracy pressure of water is 33,000 times smaller than that of electrons. If we use the mass of a water molecule to compute K'_1 instead of the mass of an electron, we find that at a density of 1 g cm^{-3} degeneracy does not set in until the temperature drops below 4 K. This does make an important point, however: for fully ionised gasses, it is much easier to be degenerate than one might think.