In the first class we discussed stars' light output and the things than can be derived directly from it – luminosity, temperature, and composition. However, to build a physical theory we require knowledge of one additional quantity: stars' masses. In this class we will discuss how those masses are derived, and begin a discussion of how we can use this information to build a physical theory for stars' structure.

I. Mass measurements using binaries

Measuring the masses of stars turns out to be surprisingly difficult – how do we measure the mass of an object sitting by itself in space? The answer turns out to be that we can't, but that we can measure the masses of objects that aren't sitting by themselves. Fortunately, nature has provided for us. Roughly 2/3 of stars in the Milky Way appear to be single stars, but the remaining 1/3 are members of multiple star systems, meaning that two or more stars are gravitationally bound together and orbit one another. Of these, binary systems, consisting of two stars are by far the most common. Binaries are important because they provide us with a method to measure stellar masses using Newton's laws alone.

As a historical aside before diving into how we measure masses, binary stars are interesting as a topic in the history of science because they represent one of the earliest uses of statistical inference. The problem is that when we see two stars close to one another on the sky, there is no obvious way to tell is the two are physically near each other, or if it is simply a matter of two distant, unrelated stars that happen to be lie near the same line of sight. In other words, just because two stars have a small angular separation, it does not necessarily mean that they have a small physical separation.

However, in 1767 the British astronomer John Michell performed a statistical analysis of the distribution of stars on the sky, and showed that there are far more close pairs than one would expect if they were randomly distributed. Thus, while Michell could not infer that any particular pair of stars in the sky was definitely a physical binary, he showed that the majority of them must be.

A. Visual binaries

Binary star systems can be broken into two basic types, depending on how we discover them. The easier one to understand is visual binaries, which are pairs of stars that are far enough apart that we can see them as two distinct stars in a telescope.

We can measure the mass of a visual binary system using Kepler's laws. To see how this works, let us go through a brief recap of the gravitational two-body problem. Consider two stars of masses M_1 and M_2. We let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the vectors describing the positions of stars 1 and 2, and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be the vector distance between them. If we set up our coordinate system so that the centre of mass is at the origin, then we know that $M_1\mathbf{r}_1 + M_2\mathbf{r}_2 = 0$. We define the reduced mass as $\mu = M_1M_2/(M_1 + M_2)$, so $\mathbf{r}_1 = -(\mu/M_1)\mathbf{r}$ and $\mathbf{r}_2 = (\mu/M_2)\mathbf{r}$.

The solution to the problem is that, when the two stars are at an angle θ in their orbit, the distance between them is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \tag{1}$$

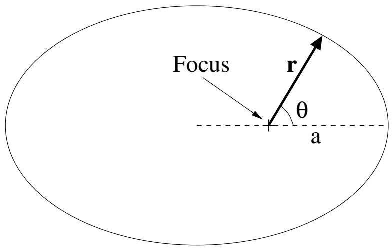

Figure 1: Graphical illustration of an orbit in the reduced two-body problem.

where the semi-major axis a and eccentricity e are determined by the stars' energy and angular momentum (Figure 1). Clearly the minimum separation occurs when $\theta = 0$ and the denominator has its largest value, and the maximum occurs when $\theta = \pi$ and the denominator takes its minimum value. The semi-major axis is the half the sum of this minimum and maximum:

$$\frac{1}{2}[r(0) + r(\pi)] = \frac{1}{2}\left[\frac{a(1-e^2)}{1+e} + \frac{a(1-e^2)}{1-e}\right] = a. \tag{2}$$

The orbital period is related to a by

$$P^2 = 4\pi^2 \frac{a^3}{GM}, \tag{3}$$

where $M = M_1 + M_2$ is the total mass of the two objects.

This describes how the separation between the two stars changes, but we instead want to look at how the two stars themselves move. The distance from each of the two stars to the centre of mass is given by

$$r_1 = \frac{\mu}{M_1} r = \frac{\mu}{M_1}\left[\frac{a(1-e^2)}{1+e\cos\theta}\right] \qquad r_2 = \frac{\mu}{M_2} r = \frac{\mu}{M_2}\left[\frac{a(1-e^2)}{1+e\cos\theta}\right]. \tag{4}$$

Again, these clearly reach minimum and maximum values at $\theta = 0$ and $\theta = \pi$, and the semi-major axes of the two ellipses describing the orbits of each star are given by half the sum of the minimum and maximum:

$$a_1 = \frac{1}{2}[r_1(0) + r_1(\pi)] = \frac{\mu}{M_1} a \qquad a_2 = \frac{1}{2}[r_2(0) + r_2(\pi)] = \frac{\mu}{M_2} a. \tag{5}$$

Note that it immediately follows that $a = a_1 + a_2$, since $\mu/M_1 + \mu/M_2 = 1$.

We can measure the mass of a visual binary using Kepler's laws. Recall that there are three laws: first, orbits are ellipses with the centre of mass of the system at one focus. Second, as the bodies orbit, the the line connecting them sweeps out equal areas in equal times – this is equivalent to conservation of angular momentum. Third, the period P of the orbit is related to its semi-major axis a by

$$P^2 = 4\pi^2 \frac{a^3}{GM}, \tag{6}$$

where M is the total mass of the two objects.

With that background out of the way, let us think about what we can actually observe, starting with the simplest case where the orbits of the binary lie in the plane of the sky, the system is close enough that we can use parallax to measure its distance, and the orbital period is short enough that we can watch the system go through a complete orbit. In this case we can directly measure four quantities,

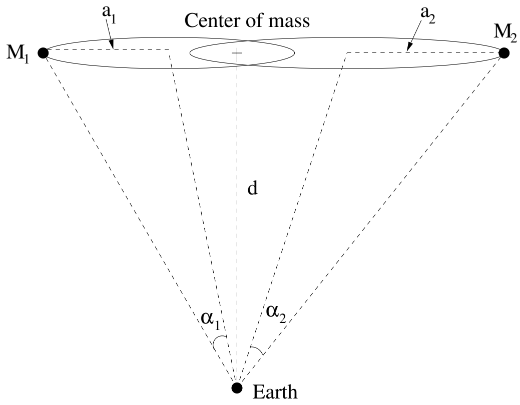

Figure 2: The motion of a visible binary whose orbit lies in the plane of the sky as seen from Earth.

which in turn tell us everything we want to know: the orbital period P, the angles subtended by the semi-major axes of the two stars orbits, α_1, and α_2, and the distance of the system, d (Figure 2).

The first thing to notice is that we can immediately infer the two stars' mass ratio just from the sizes of their orbits. The semi-major axes of the orbits are $a_1 = \alpha_1 d$ and $a_2 = \alpha_2 d$. We know that $M_1 r_1 \propto M_2 r_2$, and since $a_1 \propto r_1$ and $a_2 \propto r_2$ by the argument we just went through, we also know that $M_1 a_1 \propto M_2 a_2$. Thus it immediately follows that

$$\frac{M_1}{M_2} = \frac{a_2}{a_1} = \frac{\alpha_2}{\alpha_1}. \tag{7}$$

Note that this means we can get the mass ratio even if we don't know the distance, just from the ratio of the angular sizes of the orbits.

Similarly, we can infer the total mass from the observed semi-major axes and period using Kepler's 3rd law:

$$M = 4\pi^2 \frac{(a_1 + a_2)^3}{GP^2} = 4\pi^2 \frac{(\alpha_1 d + \alpha_2 d)^3}{GP^2}, \tag{8}$$

where everything on the right hand side is something we can observe. Given the mass ratio and the total mass, it is of course easy to figure out the masses of the individual stars. If we substitute in and write everything in terms of observables, we end up with

$$M_1 = \frac{\alpha_2}{\alpha_1 + \alpha_2} 4\pi^2 \frac{(\alpha_1 d + \alpha_2 d)^3}{GP^2} \qquad M_2 = \frac{\alpha_1}{\alpha_1 + \alpha_2} 4\pi^2 \frac{(\alpha_1 d + \alpha_2 d)^3}{GP^2}. \tag{9}$$

This is the simplest case where we see a full orbit, but in fact we don't have to wait that long – which is a good thing, because for many visual binaries the orbital period is much longer than a human lifetime! Even if we see only part of an orbit, we can make a very similar argument. All we need is to see enough of the orbit that we can draw an ellipse through it, and then we measure α_1 and α_2 for the inferred ellipse. Similarly, Kepler's second law tells us that the line connecting the two stars sweeps out equal areas in equal times, so we can infer the full orbital period just by measuring what fraction of the orbit's area has been swept out during the time we have observed the system.

The final complication to worry about is that we don't know that the orbital plane lies entirely perpendicular to our line of sight, a situation illustrated in Figure 3. In fact, we would have to be pretty lucky for this to be the case. In general we do

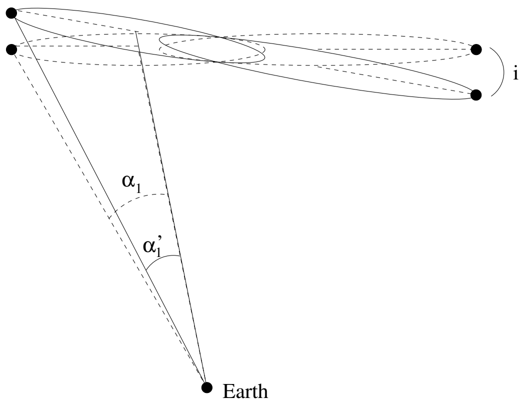

Figure 3: The motion of a visible binary whose orbit is inclined relative to the plane of the sky as seen from Earth.

not know the inclination of the orbital plane relative to our line of sight. For this reason, we do not know the angular sizes α_1' and α_2' that we measure for the orbits are different than what we would measure if the system were perfectly in the plane of the sky. A little geometry should immediately convince you that that $\alpha_1' = \alpha_1 \cos i$, where i is the inclination, and by convention $i = 0$ corresponds to an orbit that is exactly face-on and $i = 90°$ to one that is exactly edge-on. The same goes for α_2. (I have simplified a bit here and assumed that the tilt is along the minor axes of the orbits, but the same general principles work for any orientation of the tilt.)

This doesn't affect our estimate of the mass ratios, since $\alpha_1/\alpha_2 = \alpha_1'/\alpha_2'$, but it does affect our estimate of the total mass, because $a \propto \alpha_1$. Thus if we want to write out the total mass of a system with an inclined orbit, we have

$$M = 4\pi^2 \frac{(\alpha_1' d + \alpha_2' d)^3}{GP^2 \cos^3 i}. \tag{10}$$

We get stuck with a factor $\cos^3 i$ in the denominator, which means that instead of measuring the mass, we only measure a lower limit on it.

Physically, this is easy to understand: if we hold the orbital period fixed, since we can measure that regardless of the angle, there is a very simple relationship between the stars' total mass and the size of their orbit: a bigger orbit corresponds to more massive stars. However, because we might be seeing the ellipses at an angle, we might have underestimated their sizes, which corresponds to having underestimated their masses.

B. Spectroscopic binaries

A second type of binary is called a spectroscopic binary. As we mentioned earlier, by measuring the spectrum of a star, we learn a great deal about it. One thing we learn is its velocity along our line of sight – that is because motion along the line of sight produces a Doppler shift, which displaces the spectrum toward the red or the blue, depending on whether the star is moving away from us or toward us. However, we know the absolute wavelengths that certain lines have based on laboratory experiments on Earth – for example the Hα line, which is produced by hydrogen atoms jumping between the 2nd and 3rd energy states, has a wavelength of 6562.8 Å. If we see the Hα line at 6700 Å instead, we know that the star must be moving away from us.

The upshot of this is that we can use spectra to measure stars' velocity. In a binary system, we will see the velocities change over time as the two stars orbit one another. The homework assignment includes a calculation showing how it is possible to use these observations to measure the masses of the two stars.

The best case of all is when a pair of stars is observed as both a spectroscopic and a visual binary, because in that case you can figure out the masses and inclinations without needing to know the distance. In fact, it's even better than that: you can actually calculate the distance from Newton's laws!

Unfortunately, very few star systems are both spectroscopic and visual binaries. That is because the two stars have to be pretty far apart for it to be possible to see both of them with a telescope, rather than seeing them as one point of light. However, if the two stars are far apart, they will also have relatively slow orbits with relatively low velocities. These tend to produce Doppler shifts that are too small to measure. Only for a few systems where the geometry is favourable and where the system is fairly close by can we detect a binary both spectroscopically and visually. These systems are very precious, however, because then we can measure everything about them. In particular, for our purposes, we can measure their masses absolutely, with no uncertainties due to inclination or distance.

II. Hydrostatic balance

Simply by knowing stars' masses and approximate sizes, we can deduce a surprising amount about their physical state. In particular, we can show that stars must be in approximate hydrostatic balance, we can estimate their internal temperatures, and we can show that stars must be composed of gas rather than a solid or liquid.

A. The dynamical timescale

For a spherical ball of gas like a star, there are two basic forces acting on any given fluid parcel: pressure and gravity. We begin with a basic question: are these forces in balance, so that their sum must add up to zero? To answer that question, consider the counterfactual, i.e., that the two forces were significantly out of balance. What when would happen?

To answer this consider a shell of material near the surface of a star with mass M and radius R. If gravity is significantly stronger than pressure, then this material will begin to fall. If we neglect pressure entirely compared to gravity, then at the time when the shell of material has reached a radius r, assume that the mass within the shell remains the same, we can compute its velocity simply from conservation of energy. Its initial gravitational potential energy per unit mass when it reaches radius r is $-GM/r$, and if we set this equal to its kinetic energy per unit mass $v^2/2$, we have

$$v = \frac{dr}{dt} = \sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{R}}. \tag{11}$$

This is a simple ordinary differential equation for the radius r as a function of time, and it can be solved by making the trigonometric substitution $r = R\cos^2\xi$. Making this substitution gives

$$-2R\cos\xi\sin\xi\,\frac{d\xi}{dt} = \sqrt{\frac{2GM}{R}}\sqrt{\frac{1}{\cos^2\xi} - 1} \tag{12}$$

$$2\cos\xi\sin\xi\,\frac{d\xi}{dt} = \sqrt{\frac{2GM}{R^3}}\tan\xi \tag{13}$$

$$2\cos^2\xi\,d\xi = \sqrt{\frac{2GM}{R^3}}\,dt \tag{14}$$

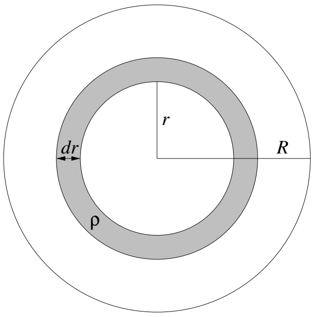

Figure 4: Schematic diagram of a shell of material within a star.

$$\xi + \frac{1}{2}\sin 2\xi \;=\; \sqrt{\frac{2GM}{R^3}}\,t. \tag{15}$$

The radius reaches $r = 0$ when $\xi = \pi/2$, so we can substitute in $\xi = /\pi/2$ to find the time when the shell of material at the outside of the star would reach the centre:

$$t \;=\; \frac{\pi}{2}\sqrt{\frac{R^3}{2GM}} \tag{16}$$

$$\;=\; 1770\left(\frac{R}{R_\odot}\right)^{3/2}\left(\frac{M}{M_\odot}\right)^{-1/2}\ \text{s}. \tag{17}$$

We refer to this time as the dynamical timescale; it is the time that would be required for the star to radically change its state if the forces acting on the material in it were out of pressure balance. For a star like the Sun, this time scale is only about half an hour. We can immediately conclude that stars like the Sun must be in exquisitely fine pressure balance, because even if they were out of balance only by a little bit, so that it took 1000 of these timescales for them to move, they would have collapse (or exploded if pressure was greater) long ago.

B. The equation of hydrostatic balance

Having established that stars should have zero net force acting on them, we are in a position to express that mathematically, and in turn to use that formula to derive a remarkable result known as the virial theorem. Consider a star of total mass M and radius R, and focus on a thin shell of material at a distance r from the star's centre. The shell's thickness is dr, and the density of the gas within it is ρ. Thus the mass of the shell is $dm = 4\pi r^2 \rho\, dr$ (Figure 4).

Let the mass interior to radius r be m. In this case the gravitational force acting on the shell is

$$F_g = -\frac{Gm\,dm}{r^2} = -4\pi G\rho m\,dr, \tag{18}$$

where the minus sign indicates that the force is inward.

The other force acting on the shell is gas pressure. Of course the shell feels pressure from the gas on either side of it, and it feels a net force only due to the difference in pressure on either side. This is just like the forces caused by air the room. The air pressure is pretty uniform, so that we feel equal force from all directions, and there is no net force in any particular direction. However, if there is a difference in pressure, there will be a net force. To calculate this, note that force is pressure times area. Thus if the pressure at the base of the shell is $P(r)$ and the pressure at its top is $P(r + dr)$, the net force that the shell feels is

$$F_p = 4\pi r^2\,[P(r) - P(r + dr)] \tag{19}$$

Note that the sign convention is chosen so that the force from the top of the shell (the $P(r+dr)$ term) is inward, and the force from the bottom of the shell is outward. In the limit $dr \to 0$, it is convenient to rewrite the pressure force in a more transparent form using the definition of the derivative:

$$\frac{dP}{dr} = \lim_{dr \to 0} \frac{P(r+dr) - P(r)}{dr}. \tag{20}$$

Substituting this into the pressure force gives

$$F_p = -4\pi r^2 \frac{dP}{dr} dr. \tag{21}$$

If there is zero net force on the stellar material, then the gravitational and pressure forces must sum to zero, and thus we must have

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}. \tag{22}$$

This equation expresses how much the pressure changes as we move through a given radius in the star, i.e., if we move upward 10 km, by how much will the pressure change? Sometimes it is more convenient to phrase this in terms of change per unit mass, i.e., if we move upward far enough so that an additional 0.01 $M_\odot$ of material is below us, how much does the pressure change. We can express this mathematically via the chain rule. The change in pressure per unit mass is

$$\frac{dP}{dm} = \frac{dP}{dr}\frac{dr}{dm} = \frac{dP}{dr}\left(\frac{dm}{dr}\right)^{-1} = \frac{dP}{dr}\frac{1}{4\pi r^2 \rho} = -\frac{Gm}{4\pi r^4}. \tag{23}$$

This is called the Lagrangian form of the equation, while the one involving dP/dr is called the Eulerian form. We will usually work with Lagrangian equations in this class.

In either form, since the quantity on the right hand side is always negative, the pressure must decrease as either r or m increase, so the pressure is highest at the star's centre and lowest at its edge. In fact, we can exploit this to make a rough estimate for the minimum possible pressure in the centre of star. We can integrate the Lagrangian form of the equation over mass to get

$$\int_0^M \frac{dP}{dm} dm = -\int_0^M \frac{Gm}{4\pi r^4} dm \tag{24}$$

$$P(M) - P(0) = -\int_0^M \frac{Gm}{4\pi r^4} dm \tag{25}$$

On the left-hand side, $P(M)$ is the pressure at the star's surface and $P(0)$ is the pressure at its centre. The surface pressure is tiny, so we can drop it. For the right-hand side, we know that r is always smaller than R, so $Gm/4\pi r^4$ is always larger than $Gm/4\pi R^4$. Thus we can write

$$P(0) \approx \int_0^M \frac{Gm}{4\pi r^4} dm > \int_0^M \frac{Gm}{4\pi R^4} dm = \frac{GM^2}{8\pi R^4} \tag{26}$$

Evaluating this numerically for the Sun gives $P_c > 4 \times 10^{14}$ dyne cm^{-2}. In comparison, 1 atmosphere of pressure is 1.0×10^6 dyne cm^{-2}, so this argument demonstrates that the pressure in the centre of the Sun must exceed 10^8 atmospheres. In fact, it is several times larger than this.

III. The Virial Theorem

We will next derive a volume-integrated form of the equation of hydrostatic equilibrium that will prove extremely useful for the rest of the class, and, indeed, is perhaps one of the most important results of classical statistical mechanics: the virial theorem. The first proof of a form of the virial theorem was accomplished by the German physicist Claussius in 1851, but numerous extensions and generalisations have been developed since. We will be using a particularly simple version of it, but one that is still extremely powerful.

A. Derivation

To derive the virial theorem, we will start by taking both sides of the Lagrangian equation of hydrostatic balance and multiplying by the volume $V = 4\pi r^3/3$ interior to some radius r:

$$V\,dP = -\frac{1}{3}\frac{Gm\,dm}{r}. \tag{27}$$

Next we integrate both sides from the center of the star to some radius r where the mass enclosed is $m(r)$ and the pressure is $P(r)$:

$$\int_0^{P(r)} V\,dP = -\frac{1}{3}\int_0^{m(r)} \frac{Gm'\,dm'}{r'}. \tag{28}$$

Before going any further algebraically, we can pause to notice that the term on the right side has a clear physical meaning. Since Gm'/r' is the gravitational potential due to the material of mass m' inside radius r', the integrand $(Gm'/r')dm'$ just represents the gravitational potential energy of the shell of material of mass dm' that is immediately on top of it. Thus the integrand on the right-hand side is just the gravitational potential energy of each mass shell. When this is integrated over all the mass interior to some radius, the result is the total gravitational potential energy of the gas inside this radius. Thus we define

$$\Omega(r) = -\int_0^{m(r)} \frac{Gm'\,dm'}{r'} \tag{29}$$

to the gravitational binding energy of the gas inside radius r.

Turning back to the left-hand side, we can integrate by parts:

$$\int_0^{P(r)} V\,dP = [PV]_0^r - \int_0^{V(r)} P\,dV = [PV]_r - \int_0^{V(r)} P\,dV. \tag{30}$$

In the second step, we dropped PV evaluated at $r' = 0$, because $V(0) = 0$. To evaluate the remaining integral, it is helpful to consider what dV means. It is the volume occupied by our thin shell of matter, i.e., $dV = 4\pi r^2\,dr$. While we could make this substitution to evaluate, it is even better to think in a Lagrangian way, and instead think about the volume occupied by a given mass. Since $dm = 4\pi r^2 \rho\,dr$, we can obviously write

$$dV = \frac{dm}{\rho}, \tag{31}$$

and this changes the integral to

$$\int_0^{V(r)} P\,dV = \int_0^{m(r)} \frac{P}{\rho}dm. \tag{32}$$

Putting everything together, we arrive at our form of the virial theorem:

$$[PV]_r - \int_0^{m(r)} \frac{P}{\rho} dm = \frac{1}{3}\Omega(r). \tag{33}$$

If we choose to apply this theorem at the outer radius of the star, so that $r = R$, then the first term disappears because the surface pressure is negligible, and we have

$$\int_0^M \frac{P}{\rho} dm = -\frac{1}{3}\Omega, \tag{34}$$

where Ω is the total gravitational binding energy of the star.

This might not seem so impressive, until you remember that, for an ideal gas, you can write

$$P = \frac{\rho k_B T}{\mu m_{\rm H}} = \frac{\mathcal{R}}{\mu}\rho T, \tag{35}$$

where μ is the mean mass per particle in the gas, measured in units of the hydrogen mass, and $\mathcal{R} = k_B/m_{\rm H}$ is the ideal gas constant. If we substitute this into the virial theorem, we get

$$\int_0^M \frac{\mathcal{R}T}{\mu} dm = -\frac{1}{3}\Omega. \tag{36}$$

For a monatomic ideal gas, the internal energy per particle is $(3/2)k_B T$, so the internal energy per unit mass is $u = (3/2)\mathcal{R}T/\mu$. Substituting this in, we have

$$\int_0^M \frac{2}{3} u \, dm = -\frac{1}{3}\Omega \tag{37}$$

$$U = -\frac{1}{2}\Omega, \tag{38}$$

where U is just the total internal energy of the star, i.e., the internal energy per unit mass u summed over all the mass in the star. This is a remarkable result. It tells us that the total internal energy of the star is simply $-(1/2)$ of its gravitational binding energy.

The total energy is

$$E = U + \Omega = \frac{1}{2}\Omega. \tag{39}$$

Note that, since $\Omega < 0$, this implies that the total energy of a star made of ideal gas is negative, which makes sense given that a star is a gravitationally bound object. Later in the course we'll see that, when the material in a star no longer acts like a classical ideal gas, the star can have an energy that is less negative than this, and thus is less strongly bound.

Incidentally, this result bears a significant resemblance to one that applies to orbits. Consider a planet of mass m, such as the Earth, in a circular orbit around a star of mass M at a distance R. The planet's orbital velocity is the Keplerian velocity

$$v = \sqrt{\frac{GM}{R}}, \tag{40}$$

so its kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2R}. \tag{41}$$

Its potential energy is

$$\Omega = -\frac{GMm}{R}, \tag{42}$$

so we therefore have

$$K = -\frac{1}{2}\Omega, \tag{43}$$

which is basically the same as the result we just derived, except with kinetic energy in place of internal energy. This is no accident: the virial theorem can be proven just as well for a system of point masses interacting with one another as we have proven it for a star, and an internal or kinetic energy that is equal to $-1/2$ of the potential energy is the generic result.

B. Application to the Sun

We'll make use of the virial theorem many times in this class, but we can make one immediate application right now: we can use the virial theorem to estimate the mean temperature inside the Sun. Let $\bar{T}$ be the Sun's mass-averaged temperature. The internal energy is therefore

$$U = \frac{3}{2}M\frac{\mathcal{R}\bar{T}}{\mu}. \tag{44}$$

The gravitational binding energy depends somewhat on the internal density distribution of the Sun, which we are not yet in a position to calculate, but it must be something like

$$\Omega = -\alpha\frac{GM}{R}, \tag{45}$$

where α is a constant of order unity that describes our ignorance of the internal density structure. Applying the virial theorem and solving, we obtain

$$\frac{3}{2}M\frac{\mathcal{R}\bar{T}}{\mu} = \frac{1}{2}\alpha\frac{GM^2}{R} \tag{46}$$

$$\bar{T} = \frac{\alpha}{3}\frac{\mu}{\mathcal{R}}\frac{GM}{R} \tag{47}$$

If we plug in $M = M_\odot$, $R = R_\odot$, $\mu = 1/2$ (appropriate for a gas of pure, ionised hydrogen), and $\alpha = 3/5$ (appropriate for a uniform sphere), we obtain $\bar{T} = 2.3 \times 10^6$ K. This is quite impressively hot. It is obviously much hotter than the surface temperature of about 6000 K, so if the average temperature is more than 2 million K, the temperature in the centre must be even hotter.

It is also worth pausing to note that we were able to deduce the internal temperature of Sun to within a factor of a few from nothing more than its bulk characteristics, and without any knowledge of the Sun's internal workings. This sort of trick is what makes the virial theorem so powerful!

We can also ask what the Sun's high temperature implies about the state of the matter in its interior. The ionisation potential of hydrogen is 13.6 eV, and for $T = 2 \times 10^6$ K, the thermal energy per particle is $(3/2)k_BT = 260$ eV. Thus the thermal energy per particle is much greater than the ionisation potential of hydrogen. Any collision will therefore lead to an ionisation, and we conclude that the bulk of the gas in the interior of a star must be nearly fully ionised.

C. Further application: which celestial objects are gasses?

Here is one more application of the virial theorem: deciding which celestial objects will have gaseous centres (like the Sun), and which will have liquid or solid centres (like the Earth). The distinguishing characteristic of a gas is that the potential energy associated with inter-particle forces at the typical particle-particle separation is small compared to the particles' thermal energy. In other words, a gas is a set of particles that are moving around at high enough speeds that the forces they exert on one another are negligible except on those rare occasions when they happen to pass extremely close to one another.

To check this for a star, consider a region of density ρ and temperature T, consisting of atoms with atomic mass $\mathcal{A}$ and atomic number $\mathcal{Z}$. The number density of the particles is $n = \rho/(\mathcal{A} m_{\mathrm{H}})$, so the typical distance between them must be

$$d = n^{-1/3} = \left(\frac{\mathcal{A} m_{\mathrm{H}}}{\rho}\right)^{1/3}.$$

The typical electromagnetic potential energy is therefore at most

$$E \simeq \frac{\mathcal{Z}^2 e^2}{d} = \mathcal{Z}^2 e^2 \left(\frac{\rho}{\mathcal{A} m_{\mathrm{H}}}\right)^{1/3},$$

where e is the electron charge. The "at most" is because this assumes that the potential energy comes from the full charge of the nuclei, neglecting any cancellation coming from electrons of opposite charge "screening" the nuclear charges.

To see how this compares to the thermal energy, i.e., to compute the ratio $E/k_B T$, ideally we would check at every point within the object since both ρ and $k_B T$ change with position. However, we can get a rough idea of what the result is going to be if we use mean values of ρ and T. For an object of mass M and radius R, $\overline{\rho} = 3M/(4\pi R^3)$, the virial theorem tells us that the mean temperature will be

$$\overline{T} = \frac{\alpha}{3}\frac{\mu}{\mathcal{R}}\frac{GM}{R} = \frac{\alpha}{3}\frac{\mathcal{A}}{\mathcal{R}}\frac{GM}{R},$$

where α is a constant of order unity, and $\mu \approx \mathcal{A}$ is the mean atomic mass per particle. (If the gas is fully ionized then μ will be lower, but we are interested in an order of magnitude result, so we will ignore this for now). Substituting $\overline{\rho}$ and $\overline{T}$, and dropping constants of order unity, we find

$$\frac{E}{k_B \overline{T}} \sim \frac{\mathcal{Z}^2 e^2}{G \mathcal{A}^{4/3} m_{\mathrm{H}}^{4/3} M^{2/3}} = 0.011 \frac{\mathcal{Z}^2}{\mathcal{A}^{4/3}} \left(\frac{M}{M_\odot}\right)^{-2/3}$$

Even for a pure iron composition, $\mathcal{Z} = 26$ and $\mathcal{A} = 56$, we have $E/k_B\overline{T} = 0.035(M/M_\odot)^{-2/3}$. This may vary some within a star, but the general result is that $E \ll k_B T$, so something with the mass of a star is essentially always going to be a gas, unless something very strange happens (which is does in some exotic cases). In contrast, if we plug in $M = M_\oplus = 6.0 \times 10^{27}$ g and consider pure iron, we get a ratio of 167 – the centre of the Earth is definitely not a gas!

More generally, if we go back to hydrogen composition, $\mathcal{Z} = \mathcal{A} = 1$, then this result suggests that $E/k_B\overline{T}$ when $M \sim 10^{-3}\ M_\odot$, or about the mass of Jupiter. Thus solid or liquid phases should be absent in bodies substantially larger than Jupiter, and begin to appear once the mass drops to that of Jupiter or less.