Our topic for today is the first of three classes about post-main sequence evolution. Today the topic is low mass stars, with the line between low and high mass to be discussed in a bit.

I. Leaving the Main Sequence

A. Main Sequence Lifetime

Stars remain on the main sequence as long as their hydrogen fuel lasts. While on the main sequence, their properties do not change much, but they do change some due to the gradual conversion of H into He. As a result of this conversion, the hydrogen mass fraction X decreases, while the helium mass fraction Y increases. As a result, the mean atomic weight changes. Recall that

$$\frac{1}{\mu_I} \approx X + \frac{1}{4}Y + \frac{1-X-Y}{\langle \mathcal{A} \rangle} \tag{1}$$

$$\frac{1}{\mu_e} \approx \frac{1}{2}(1+X) \tag{2}$$

$$\frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}. \tag{3}$$

Insterstellar gas out of which stars forms has roughly $X = 0.74$ and $Y = 0.24$ (in contrast to $X = 0.707$ and $Y = 0.274$ in the Sun, which has processed some of its H into He), which gives $\mu = 0.60$. In contrast, once all the H has been turned into He, $X = 0$ and $Y = 0.98$, which gives $\mu = 1.34$.

In stars like the Sun that are radiative in their cores, the changes occur shell by shell, so different shells have different compositions depending on their rate of burning. In stars that are convective in their cores, convection homogenises the composition of the different shells, so the entire convective core has a uniform composition.

The extent of convection makes a difference in how long it takes stars to leave the main sequence. Stars with convection in their cores do not leave the main sequence until they have converted all the mass in the convective region to He. As the convective zone fills more and more of the star, the main sequence lifetime therefore approaches the naively computed nuclear timescale $t_{\mathrm{nuc}} = \epsilon M c^2/L$.

In contrast, stars with radiative cores, like the Sun, leave the main sequence once the material in the very centre where nuclear burning occurs is converted to He. This makes their lifetimes shorter than t_{nuc}, with the minimum of $t_{\mathrm{ms}}/t_{\mathrm{nuc}}$ occurring near $1\ M_\odot$, since that is where stars are least convective. This general expectation agrees quite well with numerical results, as shown in Figure 1.

A general complication to this story is mass loss, which, for massive stars, can be significant even while they are on the main sequence. The mass loss mechanism is only generally understood. All stars have winds of gas leaving their surfaces, and these winds become more intense for more massive stars. These numerical results include a very approximate treatment of mass loss, but on the main sequence it is only significant for stars bigger than several tens of $M_\odot$.

B. Luminosity Evolution

Regardless of convection, the increase in μ results in an increase in luminosity. One can estimate this effect roughly using an Eddington model. The Eddington quartic

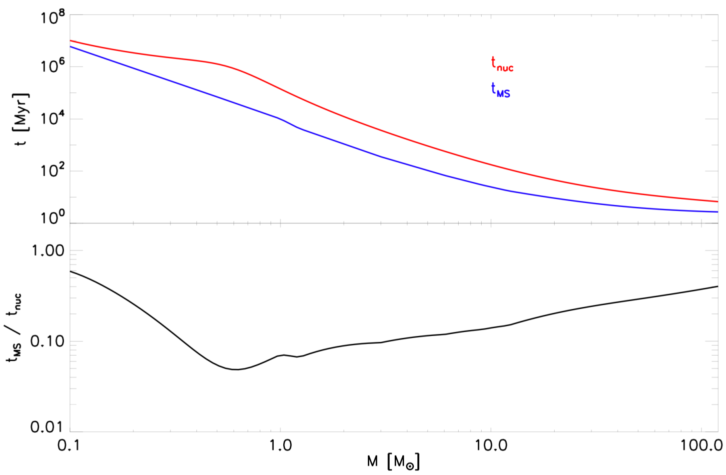

Figure 1: Top: numerically-computed main sequence lifetime (blue) and nuclear timescale (red) versus stellar mass. Bottom: the ratio of the two timescales.

is

$$0.003\left(\frac{M}{M_\odot}\right)^2\mu^4\beta^4+\beta-1=0, \tag{4}$$

and the luminosity of a star in the Eddington model is

$$\frac{L}{L_\odot}=\frac{4\pi cGM_\odot}{\kappa_s L_\odot}\mu^4\beta^4\left(\frac{M}{L_\odot}\right)^3. \tag{5}$$

Thus the luminosity at fixed mass is proportional to $(\mu\beta)^4$.

For a low mass star, the first term in the Eddington quartic is negligible, so $\beta \approx 1$ independent of μ, and thus $L \propto \mu^4$. For very massive stars the first term in the Eddington quartic dominates, which means that $\mu\beta \approx$ constant, so L stays constant. However, this will apply only to very, very massive stars. Thus in general we expect L to increase with μ, with the largest increases at low masses and smaller increases at high masses.

If an entire star were converted from H to He, this would suggest that its luminosity should go up by a factor of $(1.34/0.6)^4 = 25$ at low masses. Of course the entire star isn't converted into He except in fully convective stars that are uniform throughout, and stars are fully convective only below roughly 0.3 $M_\odot$. These stars have main sequence lifetimes larger than the age of the universe, so none have ever fully converted into He. In more massive stars that have reached the end of the main sequence, μ increases to 1.34 in their cores, but not elsewhere, so the mean value of μ and the luminosity increase by a smaller amount.

This simple understanding is also in good agreement with the results of numerical calculations. What is a bit less easy to understand analytically, but also happens, is that stars' radii swell, reducing their effective temperatures. The swelling is greatest for the most massive stars, so, although they do not move very far in L, they move a considerable distance in $T_{\rm eff}$. This effect is shown in Figure 2

II. The Red Giant Phase

A. The Schönberg-Chandrasekhar Limit

As stars reach the end of their main sequence lives, they accumulate a core of helium that is inert, in the sense that no nuclear reactions are taking place, so the core

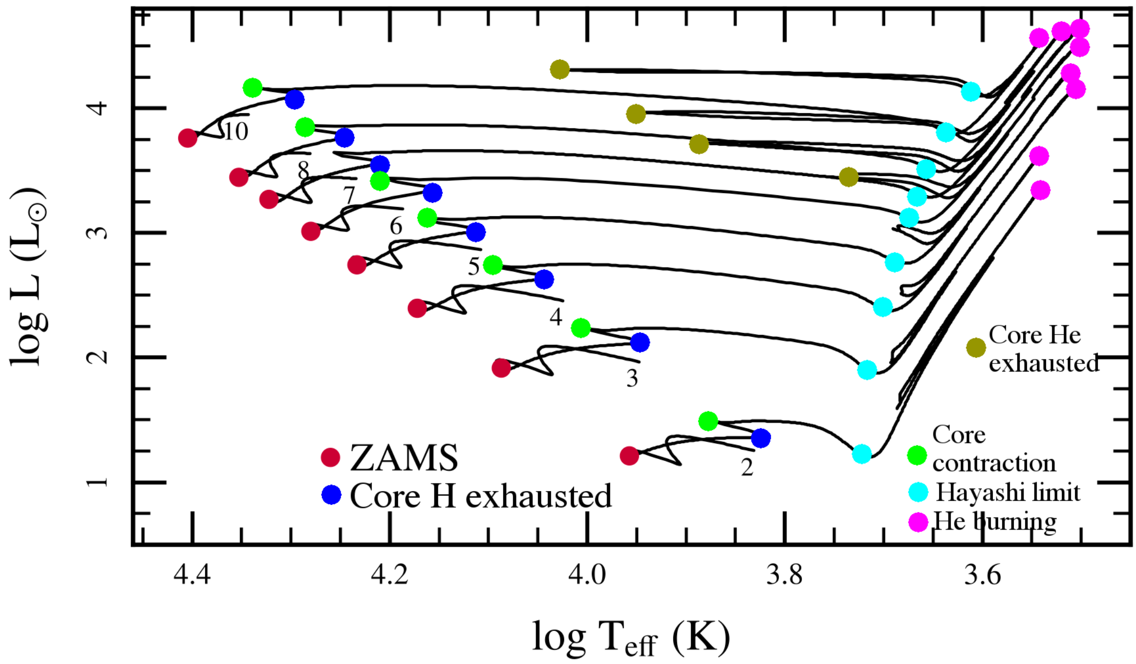

Figure 2: A numerical calculation of the evolutionary tracks of $2 - 10\ M_\odot$ stars in the HR diagram, from Paxton et al. (2011, http://adsabs.harvard.edu/abs/2011ApJS..192....3P). Stellar masses are as indicated by each track. Points on the tracks are as follows: **Red points** indicate the ZAMS. **Blue points** indicate the exhaustion of hydrogen in the core. **Green points** indicate when the core exceeds the Schönberg-Chandrasekhar limit and begins contracting. **Cyan points** indicate where the track hits the Hayashi limit. **Magenta points** indicate the onset of He burning. **Brown points** indicate the exhaustion of He. The **sub-giant branch** is between the green (core contraction) and cyan (Hayashi limit) points. The **red giant branch (RGB)** is between the cyan (Hayashi limit) and magenta (He burning) points. The **horizontal branch (HB)** is between the magenta (He burning) and brown (He exhaustion) points.

generates no energy. The consequences of this become clear if we examine the two stellar structure equations that describe energy generation and transport:

$$\frac{dL}{dm} = q_{\rm nuc} \tag{6}$$
$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{L}{(4\pi r^2)^2}. \tag{7}$$

Strictly we should write down the possibility for convective as well as radiative transport in the second equation, but we will see in a moment that is not necessary.

If there is no nuclear energy generation in the He core, then $q_{\rm nuc} = 0$, which means $dL/dm = 0$ in the core. Thus the flux through the core must be constant, and, since there is no flux emerging from $m = 0$, this means that the core must have $L = 0$. It immediately follows that $dT/dm = 0$ in the core as well – that is, the core is isothermal. This is why we do not need to worry about convection: since $dT/dr = 0$, the temperature gradient is definitely sub-adiabatic. Thus if there was any convection going on in the core, it shuts off once the nuclear reactions stop due to lack of fuel.

The star as a whole is not necessarily pushed out of thermal equilibrium by this process because nuclear burning can continue in the material above the core that still has hydrogen in it. This can be enough to power the star. However, as this material depletes its hydrogen, it too becomes inert, adding to the mass of the helium core. Thus the core grows to be a larger and larger fraction of the star as time passes.

We can show that this configuration of a growing isothermal core continue indefinitely, and, indeed, must end well before the entire star is converted to He. This point was first realized by Schönberg and Chandrasekhar in 1942, and in their honour is known as the Schönberg-Chandrasekhar limit. There are several ways to demonstrate the result, but the most straightforward is using the virial theorem.

We will apply the virial theorem to the isothermal core. It requires that

$$P_s V_c - \int_0^{M_c} \frac{P}{\rho}\, dm = \frac{1}{3}\Omega_c, \tag{8}$$

where V_c is the volume of the core, M_c is its mass, and Ω_c is its binding energy. The term P_s is the pressure at the surface of the core, and it is non-zero. This is somewhat different than when applying the virial theorem to the star as a whole: normally when we do so, we drop the surface term on the grounds that the surface pressure of a star is zero. In this case, however, the core is buried deep inside the star, so we cannot assume that the pressure on its surface is zero.

Evaluating both the integral and the term on the left hand side is easy. For the term on the right hand side, we will just use our standard approximation $\Omega_c = -\alpha G M_c^2/R_c$, where α is a constant of order unity that depends on the core's internal structure. For the integral, because the core is isothermal and of uniform composition, with a temperature T_c and a mean atomic mass $\mu_c = 1.34$, appropriate for pure helium. Assuming the core is non-degenerate (more on this in a bit), we have $P/\rho = (\mathcal{R}/\mu_c)T_c$, which is constant, so

$$P_s V_c - \frac{\mathcal{R}}{\mu_c} T_c M_c = -\frac{1}{3}\alpha \frac{G M_c^2}{R_c}. \tag{9}$$

Re-arranging this equation, we can get an expression for the surface pressure:

$$P_s = \frac{3}{4\pi}\frac{\mathcal{R}T_c}{\mu_c}\frac{M_c}{R_c^3} - \frac{\alpha G}{4\pi}\frac{M_c^2}{R_c^4}, \tag{10}$$

where we have replaced the core volume with $V_c = 4\pi R_c^3/3$.

An interesting feature of this expression is that, for fixed M_c and T_c, the pressure P_s reaches a maximum at a particular value of R_c. We can find the maximum in the usual way, by differentiating P_s with respect to R_c and solving:

$$0 = \frac{dP_s}{dR_c} = -\frac{9}{4\pi}\frac{\mathcal{R}T_c}{\mu_c}\frac{M_c}{R_c}^4 + \frac{\alpha G}{\pi}\frac{M_c^2}{R_c^5} \tag{11}$$

$$R_c = \frac{4\alpha G}{9\mathcal{R}}\frac{M_c\mu_c}{T_c}. \tag{12}$$

Plugging this in, the maximum pressure is

$$P_{s,\mathrm{max}} = \frac{3^7\mathcal{R}^4}{2^{10}\pi\alpha^3 G^3}\frac{T_c^4}{\mu_c^4 M_c^2}. \tag{13}$$

The physical meaning of this maximum is as follows: if one has a core of fixed mass and temperature, and exerts a certain pressure on its surface, it will pick a radius such that it is in equilibrium with the applied surface pressure. At low surface pressure R_c is big. In such a configuration self-gravity, represented by the term $\alpha G M_c^2/(4\pi R_c^4)$ in the equation for P_s, is unimportant compared to internal thermal pressure, represented by the term $3\mathcal{R}T_c M_c/(4\pi\mu_c R_c^3)$. As the external pressure is increased, the radius shrinks, and the thermal pressure of the core goes up as R_c^{-3}.

However, if the pressure is increased enough, the self-gravity of the core is no longer unimportant. As self-gravity grows in importance, one has to decrease the radius more and more quickly to keep up with an increase in surface pressure, because more and more of the pressure of the core goes into holding itself up against self-gravity, rather than opposing the external pressure. Eventually one reaches a critical radius where the core is exerting as much pressure on its surface as it can. Any further increase in the external pressure shrinks it further, and self-gravity gets stronger faster than the internal pressure grows. The surface pressure therefore diminishes.

The outcome of this analysis is that, if the external pressure ever exceeds the maximum we have just calculated, the core cannot be in hydrostatic equilibrium. It cannot satisfy the virial theorem. To see if this condition is met in a star with a helium core, we can estimate the pressure exerted on its surface by the rest of the star. To calculate this, we note that the envelope must obey the equation of hydrostatic equilibrium, and that we can integrate this from the surface of the isothermal core to the surface of the star:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \qquad \Longrightarrow \qquad \int_{P_s}^{0} dP = -P_s = -\int_{M_c}^{M}\frac{Gm}{4\pi r^4}\,dm. \tag{14}$$

To get a lower limit on P_s, we note that $r < R$ everywhere inside the star, so the integrand

$$\frac{Gm}{4\pi r^4} \le \frac{Gm}{4\pi R^4}. \tag{15}$$

Plugging this into the integral gives a lower limit as well:

$$P_s \leq \int_{M_c}^{M} \frac{Gm}{4\pi R^4}\, dm = \frac{G}{4\pi R^4} \int_{M_c}^{M} m\, dm = \frac{G}{8\pi R^4}\left(M^2 - M_c^2\right) \leq \frac{GM^2}{8\pi R^4}. \tag{16}$$

Since this is a lower limit on the pressure at the surface of the core, and we know that the core can only sustain a certain maximum pressure at its surface, combining this with our previous result gives a condition that the star must satisfy if it is to remain in hydrostatic equilibrium:

$$\frac{GM^2}{8\pi R^4} \leq \frac{3^7 \mathcal{R}^4}{2^{10}\pi\alpha^3 G^3} \frac{T_c^4}{\mu_c^4 M_c^2} \tag{17}$$

To see when this is likely to be violated, consider the gas just above the surface of the isothermal core. The temperature and pressure must change continuously across the core edge, so the envelope pressure and temperature there obey $T_{\rm env} = T_c$ and $P_{\rm env} = P_s$. Applying the ideal gas law to the envelope we have

$$T_{\rm env} = T_c = \frac{P_s \mu_{\rm env}}{\mathcal{R}\rho_{\rm env}}, \tag{18}$$

where $\mu_{\rm env}$ and $\rho_{\rm env}$ are the mean molecular weight and density just above the envelope. The maximum temperature occurs when P_s is at its maximum value, and substituting in $P_s = P_{s,\rm max}$ gives

$$T_c = \frac{\mu_{\rm env}}{\mathcal{R}\rho_{\rm env}} \left(\frac{3^7 \mathcal{R}^4}{2^{10}\pi\alpha^3 G^3} \frac{T_c^4}{\mu_c^4 M_c^2}\right) \tag{19}$$

$$T_c^3 = \frac{2^{10}\pi\alpha^2 G^3}{3^7 \mathcal{R}^3} \frac{\mu_c^4 M_c^4 \rho_{\rm env}}{\mu_{\rm env}} \tag{20}$$

$$\tag{21}$$

As an extremely rough estimate we can also take $\rho_{\rm env} \sim 3M/(4\pi R^3)$, and plugging this in gives

$$T_c^3 \approx \frac{2^8 \alpha^2 G^3}{3^6 \mathcal{R}^3} \frac{\mu_c^4 M_c^2 M}{\mu_{\rm env} R^3}. \tag{22}$$

Thus we have now estimate T_c terms of the properties of the star. Plugging this into our condition for stability gives

$$\frac{GM^2}{8\pi R^4} \lesssim \frac{3^7 \mathcal{R}^4}{2^{10}\pi\alpha^3 G^3} \frac{1}{\mu_c^4 M_c^2} \left(\frac{2^8 \alpha^2 G^3}{3^6 \mathcal{R}^3} \frac{\mu_c^4 M_c^2 M}{\mu_{\rm env} R^3}\right)^{4/3} \tag{23}$$

$$\frac{M_c}{M} \lesssim \sqrt{\frac{27\alpha}{512}} \left(\frac{\mu_{\rm env}}{\mu_c}\right)^2 \tag{24}$$

Doing the analysis more carefully rather than using crude approximations, the coefficient turns out to be 0.37:

$$\frac{M_c}{M} \leq 0.37 \left(\frac{\mu_{\rm env}}{\mu_c}\right)^2. \tag{25}$$

Since $\mu_{\rm env} < \mu_c$, this implies that the core can only reach some relatively small fraction of the star's total mass before hydrostatic equilibrium becomes impossible.

Using $\mu_{\mathrm{env}} = 0.6$ and $\mu_c = 1.3$, the limit is that $M_c \lesssim 0.1 M_\odot$. Once a star reaches this limit, the core must collapse.

This limit applies to stars that are bigger than about 2 $M_\odot$. For smaller stars, the gas in the He core becomes partially degenerate before the star reaches the Schönberg-Chandrasekhar limit. Since in a degenerate gas the pressure does not depend on the temperature, the pressure can exceed the result we got assuming isothermal gas. This allows the core to remain in hydrostatic equilibrium up to higher fractions of the star's mass.

B. The Sub-Giant and Red Giant Branches

Collapse of the core causes it to cease being isothermal, because it provides a new source of power: gravity. The collapse therefore allows hydrostatic equilibrium to be restored, but only at the price that the core shrinks on a Kelvin-Helmholtz timescale.

The core also heats up due to collapse, and this in turn heats up the gas around it where there is still hydrogen present. This accelerates the burning rate in the shell above the helium core. Moreover, it does so in an unstable way. The increase in temperature is driven by the KH contraction of the core, which is not sensitive to the rate of nuclear burning because none of the burning goes on in the collapsing core. Thus the burning rate will accelerate past the requirements of thermal equilibrium, and $L_{\mathrm{nuc}} > L$.

Consulting the virial theorem, we can understand what this implies must happen. Recall that we have shown several times that for stars with negligible radiation pressure support,

$$L_{\mathrm{nuc}} - L = \frac{dE}{dt} = \frac{1}{2}\frac{d\Omega}{dt} = -\frac{dU}{dt}. \tag{26}$$

Since $L_{\mathrm{nuc}} > L$, the left hand side is positive, and we conclude that Ω must increase and U must decrease. The potential energy $-\Omega \propto GM^2/R$, and the thermal energy $U \propto M\overline{T}$. Since the mass is fixed, the only way for Ω to increase is if R gets larger (since this brings Ω closer to zero), and the only way for U to decrease is for the mean temperature $\overline{T}$ to decrease.

Thus the unstable increase in nuclear burning causes the radius of the star to expand, while its mean temperature drops. In the HR diagram, this manifests as a drop in T_{eff}. As a result, the star moves to the right in the HR diagram. The phase is called the sub-giant branch, as illustrated in Figure 2.

In low mass stars the migration is slow, because the core is restrained from outright collapse by degeneracy pressure. In more massive stars the migration is rapid, since the core collapses on a KH timescale. For this reason we only see fairly low mass stars on the sub-giant branch. More massive stars cross it too rapidly for us to have any chance of finding one.

There is a limit to how red a star can get, called the Hayashi limit. This limit comes from the opacity of the star: as the surface temperature drops below ~ 4000 K, the hydrogen is all neutral, and even metals with lower ionisation potentials start to be come neutral. As a result, there are no free electrons, and the opacity drops like a rock. Consequently, the star becomes transparent. This causes the outer layers of the star to cool and fall inward, heating up again. The net effect is that there is a minimum temperature stars can reach, called the Hayash limit.

As a post-main sequence star moves to the right in the HR diagram, it eventually bumps up against this limit. Since it can no longer deal with having $L_{\rm nuc} > L$ by getting any colder at its surface, it instead has to increase its radius instead. This allows the internal temperature and the gravitational binding energy to drop, complying with energy conservation, and it also increase the luminosity, decreasing the difference between $L_{\rm nuc}$ and L. This phase of evolution is known as the red giant phase, and stars that are at low temperature and high and rising luminosity are called red giants, as shown in Figure 2.

Red giants also display an interesting phenomenon called dredge-up. The high opacity of the low-temperature envelope of the red giant guarantees that it will be convectively unstable, and the convective zone reaches all the way down to where the region where nuclear burning has taken place. It therefore drags up material that has been burned, changing the visible composition of the stellar surface. Nuclear burning destroys lithium and increases the abundance of C and N, and in red giants we can observe these altered compositions.

III. The Helium Burning Phase

We showed earlier that the temperature of the isothermal core of the star is given approximately by

$$T_c^3 \approx \frac{2^8 \alpha^2 G^3}{3^6 \mathcal{R}^3} \frac{\mu_c^4 M_c^4 \rho_{\rm env}}{\mu_{\rm env}}. \tag{27}$$

As the star ascends the red giant branch, $\rho_{\rm env}$ is dropping, but at the same time M_c is rising as more and more mass is added to the core, and its 4th power-dependence beats the first power dependence on the dropping $\rho_{\rm env}$. Thus the core heats up with time. Once it violates the Schönberg-Chandresekhar limit, and it becomes powered by gravitational contraction, it heats up even more. Thus the core is always getting hotter during the red giant phase. Eventually this can produce a new source of energy: helium burning.

A. The Triple-α Reaction

Once hydrogen has burned to helium-4, we are at the first big peak in the curve of binding energy per nucleon. The next big peak in the binding energy cure is at carbon-12, which suggests that we should expect sufficiently hot stars to burn ^{4}He to ^{12}C. However, how to actually get there is a challenge. The most obvious reaction to start is

$$^4_2\text{He} + {}^4_2\text{He} \rightleftharpoons {}^8_4\text{Be}. \tag{28}$$

However, the $^8_4$Be nucleus is violently unstable, and disintegrates in about 3×10^{-16} s. Nor can we get out of the problem by hoping for a weak reaction to convert a proton into a neutron, because there is no stable nucleus with an atomic number $\mathcal{A} = 8$.

Thus we need to jump past atomic number 8 in order to burn He. The solution to this problem was found by Edwin Salpeter in 1952. If the density and temperature get high enough, it may be possible for the $^8_4$Be nucleus to collide with another $^4_2$He nucleus before it decays. Then it will undergo the reaction

$$^8_4\text{Be} + {}^4_2\text{He} \rightarrow {}^{12}_6\text{C} + \gamma, \tag{29}$$

and arrive at carbon-12, which is stable and is another peak of binding energy per nucleon. This reaction is known as the triple-α process, because it effectively involves a three-way collision between three helium-4 nuclei, which are also known as α particles. It is not a true three-way collision, because some extra time for the third collision is provided by the lifetime of the $^8_4$Be nucleus, but it is nearly so.

In addition to the short-lived beryllium state, another factor that helps this reaction go is the existence of a resonance. It turns out that there is an excited state of the carbon-12 nucleus that coincides closely in energy with the energy of a helium-4 nucleus plus a beryllium-8 nucleus. This greatly enhances the rate at which the second step in the reaction chain takes place. Indeed, were it not for this resonance, stars would not produce significant amounts of carbon. This observation allowed Fred Hoyle to predict the existence of the ^{12}C resonance before it was actually observed, based on the argument that stars must produce carbon since carbon is observed to exist in abundance.

In environments where a significant amount of carbon builds up and the temperature is high, carbon will occasionally capture an additional helium nucleus and jump to the next peak in the binding energy curve, oxygen-16:

$$^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{16}_{8}\mathrm{O} + \gamma. \tag{30}$$

Thus stars in which the triple-α process takes place wind up containing a mixture of carbon and oxygen, with the exact ratio depending on their age, density, and temperature. Further He captures are also possible, but become increasingly unlikely as one moves up in atomic number due to the increasing Coulomb barrier.

To figure out how helium burning works in stars, we must compute the rate at which the triple-α process releases energy. The net energy released can be calculated by comparing the mass of the carbon-12 nucleus to that of three helium-4 nuclei:

$$Q = (3m_{\mathrm{He}} - m_{\mathrm{C}})c^2 = 7.28 \text{ MeV}. \tag{31}$$

The capture of a fourth He nucleus leading to oxygen-16 yields another 7.16 MeV.

To compute the reaction rate and its temperature-dependence, one can assume that there is always a small amount of $^8_4$Be by equating the creation and destruction rates – a process that we will not go through, but which yields an amount of $^8_4$Be that is roughly independent of temperature. It does not depend on temperature because the limiting factor in how much beryllium is present is the very rapid spontaneous decay of the beryllium nucleus, not the Coulomb barrier to creating it. Calculations show that the beryllium fraction is ~ 1 part in 10^{10}.

All the temperature-dependence in the reaction rate comes in the next step that of converting beryllium-8 to carbon-12. As discussed a moment ago, the reaction process for creating carbon-12 depends on a resonance. We will not go through the details of how to calculate a resonant reaction in class, but we can sketch it out briefly in order to understand the temperature-dependence of the reaction. Recall from that the reaction rate is proportional to

$$R \propto \int_0^\infty \sigma(E) E e^{-E/k_B T}\, dE. \tag{32}$$

For a non-resonant reaction, we evaluated this by using a calculation of quantum tunnelling to estimate $\sigma(E)$. For a resonant reaction, however, the process is much simpler: when there is a dominant resonance, essentially all reactions take place at energies very close to the energy required to hit the resonance. For this reason we can treat the factor Ee^{-E/k_BT} as nearly constant over the resonance, and take it out of the integral, yielding

$$R \propto E_R e^{-E_R/k_BT} \int_0^\infty \sigma(E)\, dE, \tag{33}$$

where E_R is the energy that the incoming particle must have in order to hit the resonance. Then if we let

$$\tau_R = \frac{E_R}{k_BT}, \tag{34}$$

we have

$$R \propto e^{-\tau_R} \int_0^\infty \sigma(E)\, dE. \tag{35}$$

As with the non-resonant case, all the temperature-dependence is encapsulated in the parameter τ_R, which varies as T^{-1}.

The second step in the triple-α process relies on a resonance that is at an energy $E_R = 379.5$ keV above the energy of the beryllium-8 nucleus, so that is the energy an incoming particle must have to trigger the resonance. (Note that the state in question has an energy 7.95 MeV above the ground state of carbon-12, but the relevant question is the difference between that energy and the energy of the beryllium-8 nucleus, which is much smaller.)

$$\tau_R = \frac{379.5\,\mathrm{keV}}{k_BT} = 44.0 \left(\frac{10^8\,\mathrm{K}}{T}\right). \tag{36}$$

This is normalized to 10^8 K, which is about the ignition temperature for this reaction. To go further in computing the reaction rate, we must recall that triple-α effectively requires a three-way collision. For a single particle, we said that the rate at which it encounters other particles is proportional to n. We can also view this as a probability: the probability of a collision per unit time is proportional to n. For a three-body process, we need to ask about the probability of two of them striking simultaneously or nearly so (within the 10^{-16} s lifetime of the ${}^8_4$Be nucleus. The rate at which such double-collisions occurs is proportional to the probability of one collision times the probability of another: n^2v^2. Thus we expect a collision rate that varies as n^2. We will not walk through putting this in terms of a rate coefficient, but a straightforward generalization of our existing calculation shows that the reaction rate per unit volume varies as Rn^3, while the kinetic part rate coefficient itself varies as T^{-3} – it is T^{-3} instead of the usual $T^{-3/2}$ because the collision rate varies as n^2 rather than n.

B. Evolutionary Paths of He Burning

Now that we understand how He burns, and that it will do so once the gas reaches $\sim 10^8$ K, we can use that knowledge to deduce the next stages of evolution for low mass stars.

1. Stars $\sim 1.8 - 9\ M_\odot$

First consider fairly massive stars, which turn out to be those larger than 1.8 $M_\odot$. In such stars, as they ascend the RGB, the core temperature eventually reaches $\sim 10^8$ K, which is sufficient for He burning via the 3α process. At this point He burning provides a new source of energy in the core, which halts its contraction. Burning of hydrogen continues in the shell around the He core, but, since it is no longer being driven out of equilibrium by the contraction of the He core, it slows down. This allows the star to cease expanding and instead begin to contract, and the star's luminosity to decrease. The result is that the star comes back down from the red giant branch, and moves down and to the left on the HR diagram – higher effective temperature, lower luminosity. This is shown in Figure 2.

After a short period the luminosity stabilises, and since $L_{\rm nuc} < L$, the star responds by having its envelope contract. That contraction leaves the luminosity unchanged, but moves the star to higher effective temperature. The motion is roughly horizontal in the HR diagram, so this is known as the horizontal branch.. The duration of this phase is roughly 10^8 yr, set by the amount of energy that is produced by a combination of He burning in the core and H burning in the shell. It ends when the core has been entirely transformed into C and O.

2. Stars $1 - 1.8\ M_\odot$

For stars from $1-1.8\ M_\odot$, the helium core becomes degenerate before it violates the Schönberg-Chandrasekhar limit. This does not stop it from heating up, but it does change what happens once the He ignites. Recall our discussion of runaway nuclear burning instability. In a degenerate gas, the pressure and density are not connected to the temperature. As a result, once a nuclear reaction stars it heats up the gas, but does not cause a corresponding expansion that pushes the temperature back down. This tends to cause the reaction rate to increase, leading to a runaway. This is exactly what happens in the He core of a low mass star. Once helium burning stars, it runs away, in a process called the helium flash.

The helium flash ends once the nuclear reactions generate enough energy to lift the degeneracy in the core, leading it to undergo rapid expansion. This only takes a few seconds. Thereafter, the envelope responds in a way that is essentially the opposite of what happens due to core collapse in the red giant phase: it contracts and heats up. The star therefore moves down off the red giant branch and across into the horizontal branch much like a more massive star, but it does so rapidly and violently, on a KH timescale rather than something like a nuclear timescale.

3. Stars Below $1\ M_\odot$

For an even smaller star, the core never heats up enough to reach He ignition, even once much of the core mass has been converted to He. In this case the remainder of the envelope is lost through processes that are not completely understood, and what is left is a degenerate helium core. This core then sits there and cools indefinitely. This is a helium white dwarf. Stars in this mass range therefore skip the AGB and PN phases we will discuss in a moment, and go directly to white dwarfs.

C. The AGB and PN Phases

The He burning phase ends when the core has been completely converted to carbon and oxygen. At that point, what happens is essentially a repeat of the red giant phase. The core begins to contract, driving out-of-equilibrium He burning on its surface. This forces the envelope to expand, so the star moves back to the left, and lower temperature, on the HR diagram. Once the temperature drops to ~ 4000 K at the surface, the star is up against the Hayashi limit, and the envelope cannot cool any further. Instead, the star's radius expands, leading its luminosity to rise as well. The result is that the star climbs another giant branch, this one called the asymptotic giant branch, or AGB for short.

As in the red giant phase, the cool envelope become convective, and this convection drags up to the surface material that has been processed by nuclear burning. This is called second dredge-up, and it manifests in an increase in the helium and nitrogen abundances at the surface.

While the core is contracting and the envelope is expanding, the hydrogen burning shell goes out as its temperature drops. However, contraction of the core halts once it becomes supported by degeneracy pressure. At that point the hydrogen shell reignites, and his leads to a series of unstable thermal pulses. Thermal pulses work in the following cycle. As hydrogen burns, it produces helium, which sinks into a thin layer below the hydrogen burning shell. This layer has no source of energy, so it contracts and heats up. Once it gets hot enough, it ignites, and, as we showed in the tutorial, nuclear burning in a thin shell is also unstable, because the shell can't expand fast enough to keep its temperature from rising. Thus all the accumulated He burns explosively, driving the core of the star to expand and cool, just like in the helium flash. This expansion also extinguishes the hydrogen burning. Once the He is gone, however, the cycle can resume again.

This chain of reactions and explosive burning has two other noteworthy effects. First, it temporarily produces neutron-rich environments, which synthesise elements heavier than iron by neutron capturing onto elements. Second, it briefly churns up carbon from the core and convects it to the surface. The result is that carbon appears in significant quantities on the stellar surface, producing what is known as a carbon star. This process is called third dredge-up.

AGB stars also have significant stellar winds, which drive large amounts of mass loss from them. The details are not at all understood, but observationally we know that mass loss rates can reach $\sim 10^{-4}\ M_{\odot}\ \mathrm{yr}^{-1}$. The mechanism responsible for carrying the winds is likely radiation pressure, which is very significant in these stars due to their high luminosities. These winds carry lots of carbon with them, which condenses as the gas moves away from the stars and produces carbonaceous dust grains in interstellar space. The winds also reduce the total mass of the star significantly.

The winds eventually remove enough mass from the envelope that all nuclear burning there ceases, and the star finally goes out. However, the core remains very hot, and, once enough mass is removed, it is directly exposed and shines out the escaping gas. The high energy photons produced by the hot core surface are sufficient to ionize this gas, and the entire ejected shell of material lights up like a Christmas tree. This object is known as a planetary nebula. (Even though it has nothing to do with planets, the people who named it didn't know that at the time, and through a very low resolution telescope they look vaguely planetary.)

PN are some of the most visually spectacular objects in the sky, due to the variety of colours produced by the ionised gas, and the complex shapes whose origins we do not understand, as shown in Figure 3.

IV. White Dwarfs

The final state once the gas finishes escaping is a degenerate core of carbon and oxygen

Figure 3: The Cat's Eye Nebula, a planetary nebula, as seen in a combination of X-ray emission (blue) and optical emission (green and red). From https://en.wikipedia.org/wiki/Planetary_nebula.

with a typical mass of $\sim 0.6\ M_\odot$. Lower mass stars that cannot ignite helium end up with masses of $\sim 0.2 - 0.4\ M_\odot$. We can understand the final evolution of these stars with a simple model. The center of the star consists of a degenerate electron gas. However, the pressure must go to zero at the stellar surface, so at some radius the pressure and density must begin to drop, and the gas ceases to be degenerate. Thus the star consists of a degenerate core containing most of the mass, and a non-degenerate envelope on top of it. Within the degenerate part, thermal conductivity is extremely high, so the gas is essentially isothermal – it turns out that a degenerate material acts much like a metal, and conducts very well.

In the non-degenerate part of the star, the standard equations of hydrostatic balance and radiative diffusion apply:

$$\frac{dP}{dr} = -\rho \frac{GM}{r^2} \tag{37}$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}. \tag{38}$$

Note that we have M and not m in the numerator of the hydrostatic balance equation because we're approximating that all of the star's mass is in the inner, degenerate part. We also approximate that all the energy lost from the star comes from the inner, degenerate part, so $L =$ constant in the non-degenerate layer. Finally, note that the energy conservation equation $dl/dm = q$ does not apply, because we are not assuming that the star is in thermal equilibrium – indeed, it cannot be without a source of nuclear energy.

We assume that the opacity in the non-degenerate part of the star is a Kramer's opacity

$$\kappa = \kappa_0 \rho T^{-7/2} = \frac{\kappa_0 \mu}{\mathcal{R}} P T^{-9/2}, \tag{39}$$

where we have used the ideal gas law to set $\rho = (\mu/\mathcal{R})(P/T)$. Substituting this into the radiative diffusion equation gives

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{1}{T^3}\left(\frac{\kappa_0\mu}{\mathcal{R}} P T^{-9/2}\right)\rho \frac{L}{4\pi r^2} = -\frac{3\kappa_0\mu}{16\pi ac\mathcal{R}}\frac{P\rho}{T^{15/2}}\frac{L}{r^2}. \tag{40}$$

If we now divide by the equation of hydrostatic balance, we obtain

$$
\begin{aligned}
\frac{dT}{dP} &= \frac{3\kappa_0\mu}{16\pi ac\mathcal{R}G}\frac{P}{T^{15/2}}\frac{L}{M} && (41) \\
P\,dP &= \frac{16\pi ac\mathcal{R}G}{3\kappa_0\mu}\frac{M}{L}T^{15/2}\,dT. && (42)
\end{aligned}
$$

We can integrate from the surface, where $P = 0$ and $T = 0$ to good approximation, inward, and obtain the relationship between pressure and temperature

$$
\int_0^P P'\,dP' = \frac{16\pi ac\mathcal{R}G}{3\kappa_0\mu}\frac{M}{L}\int_0^T T'^{15/2}\,dT' \quad\Longrightarrow\quad P = \left(\frac{64\pi ac\mathcal{R}G}{51\kappa_0\mu}\right)^{1/2}\left(\frac{M}{L}\right)^{1/2}T^{17/4}. \tag{43}
$$

Using the ideal gas law $\rho = (\mu/\mathcal{R})(P/T)$ again, we can turn this into

$$
\rho = \left(\frac{64\pi ac\mu G}{51\kappa_0\mathcal{R}}\right)^{1/2}\left(\frac{M}{L}\right)^{1/2}T^{13/4}. \tag{44}
$$

This relationship between density and temperature must hold everywhere in the ideal gas region, and so we can apply it at the boundary between that region and the degenerate region. The pressure in the non-degenerate region is just

$$
P_{\rm nd} = \frac{\mathcal{R}}{\mu_e}\rho T, \tag{45}
$$

where we've used $\mu = \mu_e$ because the electron pressure completely dominates. Just on the other side of the boundary, in the degenerate region, the pressure is

$$
P_{\rm d} = K_1'\left(\frac{\rho}{\mu_e}\right)^{5/3}. \tag{46}
$$

Pressure, density, and temperature must change continuously across the boundary, so the ρ that appears in these two expressions is the same. Moreover, since the core is isothermal, $T = T_c$, where T_c is the core temperature. Finally, since the pressures must match across the boundary, we have

$$
\begin{aligned}
\frac{\mathcal{R}}{\mu_e}\rho T_c &= K_1'\left(\frac{\rho}{\mu_e}\right)^{5/3} && (47) \\
T &= \frac{K_1'}{\mathcal{R}\mu_e^{2/3}}\rho^{2/3} && (48) \\
&= \frac{K_1'}{\mathcal{R}\mu_e^{2/3}}\left[\left(\frac{64\pi ac\mu G}{51\kappa_0\mathcal{R}}\right)^{1/2}\left(\frac{M}{L}\right)^{1/2}T^{13/4}\right]^{2/3} && (49) \\
\frac{L}{M} &= \frac{64\pi acGK_1'^3\mu}{51\mathcal{R}^4\kappa_0\mu_e^2}T_c^{7/2}. && (50)
\end{aligned}
$$

We have therefore derived the luminosity of a white dwarf in terms of the temperature of its degenerate core. Plugging in typical values gives

$$
\frac{L/L_\odot}{M/M_\odot} \approx 6.8\times 10^{-3}\left(\frac{T_c}{10^7\ \mathrm{K}}\right)^{7/2}. \tag{51}
$$

We can use this relation to infer how long white dwarfs will shine brightly enough for us to see them. The internal energy of the white dwarf is just the thermal energy of the gas. Since the electrons are degenerate they cannot lose energy – there are no lower energy states available for them to occupy. The ions, however, are not degenerate, and they can cool off. Since the ions are a non-degenerate ideal gas, their internal energy is

$$U_I = \frac{3}{2}\frac{\mathcal{R}}{\mu_I} M T_c, \tag{52}$$

and conservation of energy requires that

$$L = -\frac{dU_I}{dt} = -\frac{3}{2}\frac{\mathcal{R}}{\mu_I} M \frac{dT_c}{dt}. \tag{53}$$

It is convenient to recast this relation in terms of the luminosity. Using our temperature-luminosity relationship we have

$$T_c = \left(\frac{51\mathcal{R}^4\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\frac{L}{M}\right)^{2/7} \tag{54}$$

$$\frac{dT_c}{dt} = \frac{2}{7}\left(\frac{51\mathcal{R}^4\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\frac{1}{M}\right)^{2/7} L^{-5/7}\frac{dL}{dt} \tag{55}$$

Plugging this into the equation for L gives

$$L = -\frac{3}{7}\frac{\mathcal{R}^{15/7}}{\mu_I} M^{5/7}\left(\frac{51\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\right)^{2/7} L^{-5/7}\frac{dL}{dt}. \tag{56}$$

Separating the variables and integrating from an initial luminosity L_0 to a luminosity L at some later time, we have

$$\int_{L_0}^{L} L'^{-12/7}\, dL' = -\frac{7}{3}\frac{\mu_I}{\mathcal{R}^{15/7}} M^{-5/7}\left(\frac{51\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\right)^{-2/7}\int_0^t dt' \tag{57}$$

$$-\frac{7}{5}\left(L^{-5/7} - L_0^{-5/7}\right) = -\frac{7}{3}\frac{\mu_I}{\mathcal{R}^{15/7}} M^{-5/7}\left(\frac{51\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\right)^{-2/7} t \tag{58}$$

$$L = L_0\left[1 + \frac{5}{3}\frac{\mu_I}{\mathcal{R}^{15/7}}\left(\frac{L_0}{M}\right)^{5/7}\left(\frac{51\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\right)^{-2/7} t\right]^{-7/5} \tag{59}$$

For long times t, we can drop the $+1$, and we find that $L \propto t^{-7/5}$. Since the white dwarf birthrate in the galaxy is about constant, this immediately yields an important theoretical prediction. The number of white dwarfs we see with a given luminosity should be proportional to the amount of time they spend with that luminosity, which we have just shown varies as $t \propto L^{-5/7}$. Thus luminous white dwarfs should be rare because they cool quickly, while dimmer ones should be more common because they cool more slowly, and the ratio of the number of white dwarfs with luminosity L_1 to the number with luminosity L_2 should vary as $(L_1/L_2)^{-5/7}$. Observations confirm this result.

We can also define a characteristic cooling time $t_{\rm cool}$ as the time it takes a white dwarf's luminosity to change significantly. This is simply the time required for the second term in parentheses to become of order unity, which is

$$t_{\rm cool} \approx \frac{3\mathcal{R}^{15/7}}{5\mu_I}\left(\frac{51\kappa_0\mu_e^2}{64\pi acGK_1'^3\mu}\right)^{2/7}\left(\frac{M}{L_0}\right)^{5/7} \approx 2.5\times 10^6\left(\frac{M/M_\odot}{L/L_\odot}\right)^{5/7}\ \text{yr}. \tag{60}$$

Thus we conclude that white dwarfs with luminosities of $L \sim 10^4 L_\odot$, typical of the planetary nebula phase, should last only a few thousand years, while those with much lower luminosities $\sim L_\odot$ can remain at that brightness for of order a million years.