Problem set 2
Due Thursday, 23 March, 2017

1. **Temperature Profiles of Stars. [20 points]**
   Consider a star with a density distribution $\rho = \rho_0(R/r)$, where R is the star's outer radius. The star's luminosity is L, and all of its energy is generated in a small region near $r = 0$. Outside that region the luminosity $L(m)$ transmitted through each mass shell is constant.

   (a) [**5 points**] Find the surface temperature of the star T_s.

   (b) [**5 points**] Assuming the opacity is dominated by electron scattering at all radii, solve for the temperature as a function of radius inside the star, excluding the energy-generating region. (Hint: the algebra will be easier if you rewrite the luminosity in terms of T_s.)

   (c) [**5 points**] Repeat the calculation from part (b) assuming the opacity is described by a Kramers law approximation with $\kappa = \kappa_0 \rho T^{-7/2}$.

   (d) [**5 points**] Make a plot of $\log(T/T_s)$ versus $\log(r/R)$ for your answers to parts (b) and (c), using the numerical values $\kappa_{\rm es} = 0.34$ cm^2 g^{-1}, $\kappa_0 \rho_0 T_s^{-7/2} = 10^8$ cm^2 g^{-1}, and $\rho_0 R = 10^{11}$ cm.

2. **The Brown Dwarf Boundary. [10 points]**
   Consider a star of Solar composition with density profile $\rho = \rho_c(1 - r/R)$. As part of problem 3 on problem set 1, we computed the central density and pressure, ρ_c and P_c in terms of M and R.

   (a) [**5 points**] Assuming that the gas at the centre of the star is non-degenerate, non-relativistic, and that radiation pressure is negligible, derive a maximum central temperature for a given mass.

   (b) [**5 points**] "Stars" with masses below a certain value never reach central temperatures high enough to start burning hydrogen; such objects are called brown dwarfs. Estimate the mass that separates stars from brown dwarfs, assuming H burning begins at $T = 10^7$ K. A more realistic density distribution puts this boundary at around 0.075 $M_\odot$; how does this compare to your simple estimate?

3. **Classical Nuclear Reactions. [20 points]**
   In this problem we will make crude estimates for nuclear reaction rates using classical mechanics, in order to demonstrate why quantum effects are necessary to explain nuclear energy generation in the Sun. We will consider the first reaction in the *pp*-chain,

   $$p + p \rightarrow {}^2\mathrm{D} + e^+ + \nu.$$

   (a) [**5 points**] What is the Coloumb (electromagnetic) potential energy per proton when the two protons are separated by a distance r? Assuming that the strong nuclear force causes the potential to become negative for $r < r_0 \simeq 2$ fm, what is the maximum potential energy? This is called the Coulomb barrier. Express your result in MeV.

   (b) [**5 points**] The rate at which a proton experiences collisions with other protons is $n\sigma v$, where n is the number density, σ is the cross-section, and v is the proton velocity. Take v to be the mean proton velocity, and compute σ assuming that the protons must get within a distance r_0 of one another to react. Estimate the collision rate using conditions appropriate to the centre of the Sun: proton density $n \simeq 10^{26}$ cm^{-3} and temperature $T = 1.6 \times 10^7$ K.

(c) **[5 points]** The rate at which collisions penetrate the Coulomb barrier is roughly $n\sigma v P_{\text{penetrate}}$, where, in the absence of quantum tunnelling, $P_{\text{penetrate}}$ is the probability that the energy of the collision is sufficient for the protons to penetrate the Coulomb barrier. Compute this probability, assuming that the protons follow a Maxwellian velocity distribution

$$n(E)\,dE = \frac{2n}{\pi^{1/2}(k_B T)^{3/2}} E^{1/2} e^{-E/k_B T}\,dE,$$

where E is the proton energy. (Hint: integrate by parts and make some substitutions, and then use the approximation that

$$\int_{x_0}^{\infty} e^{-x^2}\,dx \approx \frac{e^{-x_0^2}}{\pi^{1/2} x_0}$$

for $x_0 \gg 1$.)

(d) **[5 points]** Using your result from part (c), estimate the reaction rate for a proton at the centre of the Sun. Given this reaction rate, and the fact that there are $\sim M_\odot/m_p$ protons in the Sun, how many nuclear reactions could have occurred during the Sun's $\sim 10^{10}$ yr lifetime if there were no quantum tunnelling?

4. **Convection with Composition Gradients. [20 points]**
In deriving the stability condition and the Brunt-Väisäla frequency for convective motions, we assumed that the mean molecular weight μ was constant. However, in evolved stars there can be composition gradients (due to differing amounts of nuclear burning as a function of radius), so that μ varies with r. Composition gradients are also present in other convective systems, such as planetary atmospheres and the Earth's ocean, where salinity and thus mean molecular weight varies with depth. In this problem we will extend our analysis of convective stability to gas with a composition gradient.

(a) **[10 points]** Derive the adiabatic temperature gradient $(dT/dr)_{\text{ad}}$ in an ideal gas where the composition varies with radius, i.e., where $d\mu/dr \neq 0$. Give your answer in terms of g and $d\mu/dr$.

(b) **[10 points]** As for a system with uniform composition, the gas is stable against convection only if $dT/dr > (dT/dr)_{\text{ad}}$. In most situations $d\mu/dr < 0$, since material of lower particle mass tends to be found at higher altitude. Does a value of $d\mu/dr > 0$ make a system more or less stable against convection? In other words, is the temperature gradient required to start convection shallower or steeper in a system with $d\mu/dr > 0$ than in one with $d\mu/dr = 0$? Give a physical interpretation of your answer, as well as a mathematical justification.

5. **Luminosity Variation from Stellar Pulsation. [ASTR 4007/6007 only; 15 points]**
In this problem we will make a crude estimate of how the luminosity variation of a pulsating star is related to the variation in its radius. This provides a way of estimating the amount by which the radius is changing for a star that we observe to pulse. We will consider a star whose unperturbed state consists of a luminosity L_0, a radius R_0, and a surface temperature T_0.

(a) **[5 points]** The star's luminosity is related to its radius and surface temperature by $L = 4\pi R^2 \sigma T^4$. Suppose that, as a result of pulsation, the radius changes by an amount δR and the surface temperature changes by an amount δT. Estimate the resulting change in luminosity δL. You may assume that δR, δT, and δL are all small, so the equations can be linearised.

(b) **[5 points]** Assume that the star is composed of an adiabatic, ideal gas with adiabatic index γ_a, and that the expansion and contraction of the star are homologous. Using these assumptions, derive a relationship between δR and δT.

(c) **[5 points]** Use the result of part (b) to eliminate δT from your answer to part (a), and arrive at an estimate for the relationship between δL and δR. Based on this result, does the peak luminosity of a pulsating star with $\gamma_a = 5/3$ occur when its radius is at its maximum value or its minimum value?