Radiation from a Moving Charge

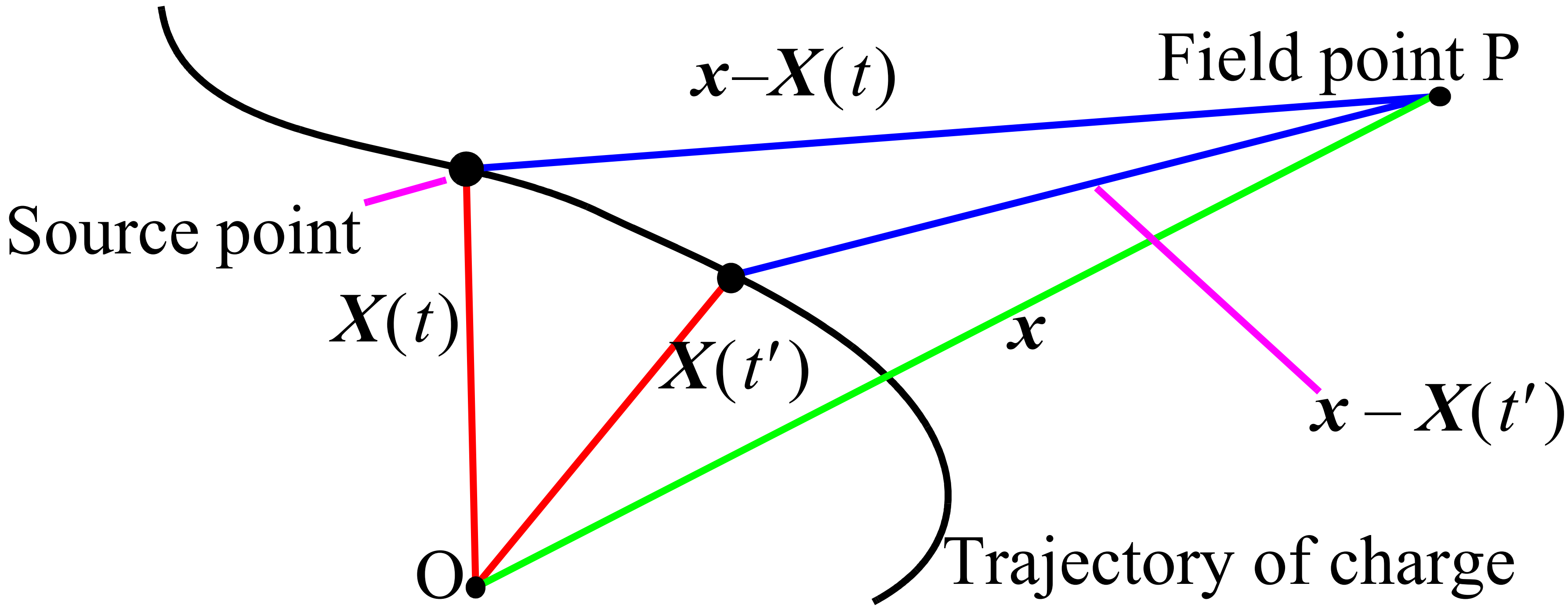
1 The Lienard-Weichert radiation field

For details on this theory see the accompanying background notes on electromagnetic theory (EM_theory.pdf)

Much of the theory in this chapter can be applied to many cases of radiating charges. Its main use will be in the application to synchrotron radiation.

Other reading: Rybicki & Lightman, ***Radiative Processes in Astrophysics***, on which the following development is mainly, but not entirely, based. In particular, the emissivity in the Stokes parameters is not dealt with in R&L.

1.1 Retarded time and position



In the above diagram:

$$
\begin{aligned}
\mathrm{O} &= \text{Arbitrary coordinate origin} \\
\mathrm{P} &= \text{Field point} \\
t &= \text{time} \\
t' &= \text{Retarded time} \\
\boldsymbol{x} &= \text{Position vector of field point} \\
\boldsymbol{X}(t) &= \text{Position vector of moving charge (source point)} \\
\boldsymbol{X}(t') &= \text{Position vector of moving charge at retarded time}
\end{aligned}
\tag{1}
$$

Retarded time and retarded position vector

The retarded time is defined as the time of emission of an electromagnetic wave from the particle which arrives at the field point at time t, i.e.

$$\begin{aligned} t &= t' + \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c} \\ \Rightarrow t' &= t - \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c} \end{aligned} \tag{2}$$

The retarded ***position vector*** is:

$$\boldsymbol{r}' = \boldsymbol{x} - \boldsymbol{X}(t') \tag{3}$$

The corresponding ***unit vector*** pointing from the retarded source point to the field point is:

$$\boldsymbol{n}' = \frac{\boldsymbol{x} - \boldsymbol{X}(t')}{|\boldsymbol{x} - \boldsymbol{X}(t')|} \tag{4}$$

Velocities

$$\text{Velocity of charge} = \dot{\boldsymbol{X}}(t)$$

$$\beta(t) = \frac{\dot{\boldsymbol{X}}(t)}{c} \tag{5}$$

$$\beta(t') = \frac{\dot{\boldsymbol{X}}(t')}{c}$$

1.2 The scalar and vector potential (Lienard- Weichert potentials)

$$\text{Scalar potential} = \phi(t, \boldsymbol{x}) = \left(\frac{q}{4\pi\varepsilon_0 r'}\right)\frac{1}{[1 - \beta(t') \cdot \boldsymbol{n}']}$$

$$\text{Vector potential} = \boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0 q}{4\pi r'}\frac{\dot{\boldsymbol{X}}(t')}{[1 - \beta(t') \cdot \boldsymbol{n}']} \tag{6}$$

Electromagnetic field

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \text{curl } \boldsymbol{A} \tag{7}$$

The result of the differentiations is:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x}, t) &= \frac{q}{4\pi\varepsilon_0 r'^2}(1-\beta'\cdot\boldsymbol{n}')^{-3} \times \\ &\quad \times \left[(\boldsymbol{n}'-\beta')\left(1-\beta'^2+\frac{r'\dot{\beta}\cdot\boldsymbol{n}'}{c}\right)-\frac{r'\dot{\beta}'(1-\beta'\cdot\boldsymbol{n}')}{c}\right] \\ \boldsymbol{B} &= c^{-1}(\boldsymbol{n}'\times\boldsymbol{E})\end{aligned} \tag{8}$$

Many of the terms in these expressions decrease as r'^{-2}. These correspond the Coulomb field of the moving charge. However, the terms proportional to the acceleration only decrease as r'^{-1}. These are the radiation terms:

$$E_{\mathrm{rad}} = \frac{q}{4\pi c\varepsilon_0 r'} \frac{[(\boldsymbol{n}' - \beta')\dot{\beta}' \cdot \boldsymbol{n}' - \dot{\beta}'(1 - \beta' \cdot \boldsymbol{n}')]}{(1 - \beta' \cdot \boldsymbol{n}')^3} \tag{9}$$

Note that

$$E_{\mathrm{rad}} \cdot \boldsymbol{n}' = 0 \tag{10}$$

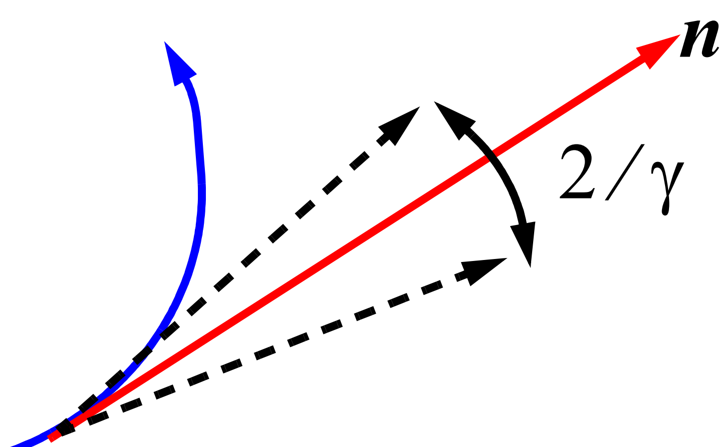
Poynting flux

The Poynting flux of the Lienard-Weichert electromagnetic field is given by:

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} = \frac{\boldsymbol{E} \times (\boldsymbol{n}' \times \boldsymbol{E})}{c\mu_0} = c\varepsilon_0[E^2\boldsymbol{n}' - (\boldsymbol{E} \cdot \boldsymbol{n}')\boldsymbol{E}] \tag{11}$$

$$= c\varepsilon_0 E^2 \boldsymbol{n}' \qquad \text{for the radiation component}$$

This can be understood in terms of equal amounts of electric and magnetic energy density ($(\varepsilon_0/2)E^2$) moving at the speed of light in the direction of $\boldsymbol{n}'$. This is a very important expression when it comes to calculating the spectrum of radiation emitted by an accelerating charge.



Trajectory of particle

Illustration of the beaming of radiation from a relativistically moving particle.

2 Radiation from relativistically moving charges – Relativistic beaming

Note the factor $(1-\beta'\cdot\boldsymbol{n}')^{-3}$ in the expression for the electric field. Expressing

$$1-\beta'\cdot\boldsymbol{n}' = 1-\beta'\cos\theta \qquad (12)$$

where θ is the angle between β' and $\boldsymbol{n}'$, we can see that this factor is small when

- $\beta' \sim 1$ – the particle is relativistic
- $\theta \approx 0$ – the field point is in the direction of the particle

When $1 - \beta' \cdot \boldsymbol{n}' \approx 0$ the contribution to the electric field is large because of the factor $(1 - \beta' \cdot \boldsymbol{n}')^{-3}$ in the expression for the electric vector.

We quantify this further as follows:

$$
\begin{aligned}
1 - \beta' \cdot \boldsymbol{n}' &= 1 - |\beta'| \cos\theta \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{1}{2}\theta^2\right) \\
&= 1 - \left(1 - \frac{1}{2\gamma^2} - \frac{\theta^2}{2}\right) \\
&= \frac{1}{2\gamma^2} + \frac{\theta^2}{2} = \frac{1}{2\gamma^2}(1 + \gamma^2\theta^2)
\end{aligned}
\tag{13}
$$

The minimum value of $1-\beta' \cdot \boldsymbol{n}'$ is $1/(2\gamma^2)$ and the value of this quantity only remains near this for $\theta \sim 1/\gamma$. This means that the radiation from a moving charge is beamed into a narrow cone of angular extent $1/\gamma$. This is particularly important in the case of synchrotron radiation for which $\gamma \sim 10^4$ (and higher) is often the case.

3 The spectrum of a moving charge

3.1 Fourier representation of the field

Consider the transverse electric field, $\boldsymbol{E}(t)$, resulting from a moving charge, at a point in space and represent it in the form:

$$\boldsymbol{E}(t) = E_1(t)\boldsymbol{e}_1 + E_2(t)\boldsymbol{e}_2 = E_\alpha(t)\boldsymbol{e}_\alpha \qquad \alpha = 1, 2 \qquad (14)$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are appropriate axes in the plane of the wave. (Note that in general we are <u>not</u> dealing with a monochromatic wave, here.)

We approach the problem of determining the spectrum by using Fourier analysis.

The Fourier transform relations for the electric field:

$$
\begin{aligned}
E_\alpha(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} E_\alpha(t)\,dt \\
E_\alpha(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} E_\alpha(\omega)\,d\omega
\end{aligned}
\qquad (15)
$$

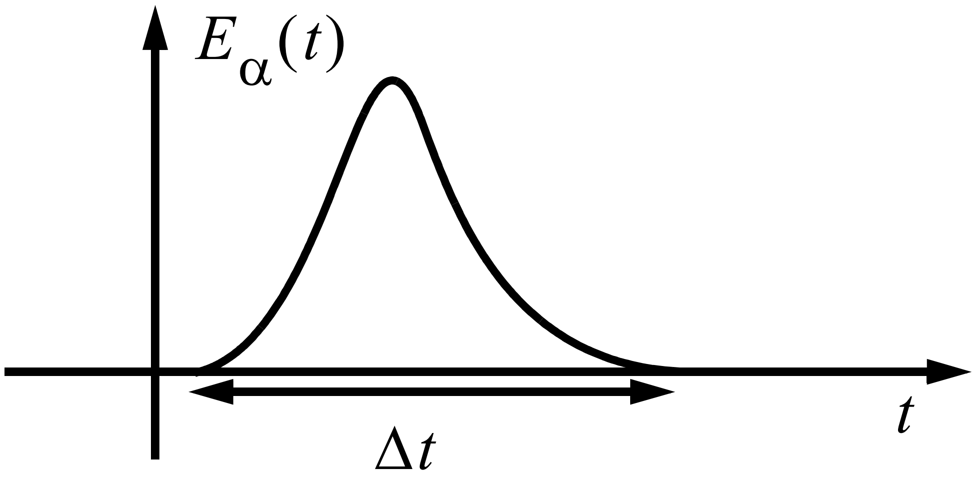
The condition that $E_\alpha(t)$ be real is that

$$
E_\alpha(-\omega) = E_\alpha^*(\omega) \qquad (16)
$$

Note: We do not use a different symbol for the Fourier transform. The transformed variable is indicated by its argument.

3.2 Spectral power in a pulse

Outline of the following calculation



Diagrammatic representation of a pulse of radiation with a duration Δt.

- Consider a pulse of radiation
- Calculate total energy per unit area in the radiation.
- Use Fourier transform theory to calculate the spectral distribution of energy.
- Show that this can be used to calculate the spectral *power* of the radiation.

Nitty-gritty

The energy per unit time per unit area of a pulse of radiation is given by:

$$\frac{dW}{dt\,dA} = \text{Poynting Flux} = (c\varepsilon_0)E^2(t)$$
$$= (c\varepsilon_0)[E_1^2(t) + E_2^2(t)] \qquad (17)$$

where E_1 and E_2 are the components of the electric field wrt (so far arbitrary) unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the plane of the wave. We refer to the two components of the transverse electromagnetic wave as different ***modes*** of polarisation.

Note that there is a difference here from the Poynting flux for a pure monochromatic plane wave in which we pick up a factor of $1/2$. That factor results from the time integration of $\cos^2\omega t$ which comes from, in effect, $\int_0^\infty |E_\alpha(\omega)|^2 d\omega$. This factor, of course, is not evaluated here since the pulse has an arbitrary spectrum.

The total energy per unit area in the α–component of the pulse is

$$\frac{dW_{\alpha\alpha}}{dA} = (c\varepsilon_0)\int_{-\infty}^{\infty} E_\alpha^2(t)dt \qquad (18)$$

(The reason for the $\alpha\alpha$ subscript is evident below.)

From Parseval's theorem,

$$\int_{-\infty}^{\infty} E_\alpha^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |E_\alpha(\omega)|^2 d\omega \tag{19}$$

The integral from $-\infty$ to ∞ can be converted into an integral from 0 to ∞ using the reality condition: For the negative frequency components, we have

$$E_\alpha(-\omega) \times E_\alpha^*(-\omega) = E_\alpha^*(\omega) \times E_\alpha(\omega) = |E_\alpha(\omega)|^2 \tag{20}$$

so that

$$\int_{-\infty}^{\infty} E_\alpha^2(t)dt = \frac{1}{\pi}\int_{0}^{\infty} |E_\alpha(\omega)|^2 d\omega \tag{21}$$

The total energy per unit area in the pulse, associated with the α component, is

$$\frac{dW_{\alpha\alpha}}{dA} = c\varepsilon_0 \int_{-\infty}^{\infty} E_\alpha^2(t)\,dt = \frac{c\varepsilon_0}{\pi} \int_0^{\infty} \left|E_\alpha(\omega)\right|^2 d\omega \qquad (22)$$

We identify the spectral components of the contributors to the Poynting flux by:

$$\frac{dW_{\alpha\alpha}}{d\omega\, dA} = \frac{c\varepsilon_0}{\pi} \left|E_\alpha(\omega)\right|^2 \qquad (23)$$

The quantity $\frac{dW_{\alpha\alpha}}{d\omega dA}$ represents the total energy per unit area per unit circular frequency in the *entire pulse,* i.e. we have accomplished our aim and determined the *spectrum* of the pulse.

We can use this expression to evaluate the power associated with the pulse. Suppose the pulse repeats with period T, then we define the power associated with component α by:

$$\frac{dW_{\alpha\alpha}}{dA d\omega dt} = \frac{1}{T}\frac{dW}{dA d\omega} = \frac{c\varepsilon_0}{\pi T}|E_\alpha(\omega)|^2 \tag{24}$$

This is equivalent to integrating the pulse over, say several periods and then dividing by the length of time between pulses.

3.3 Emissivity

The emissivity is defined in the folowing way:

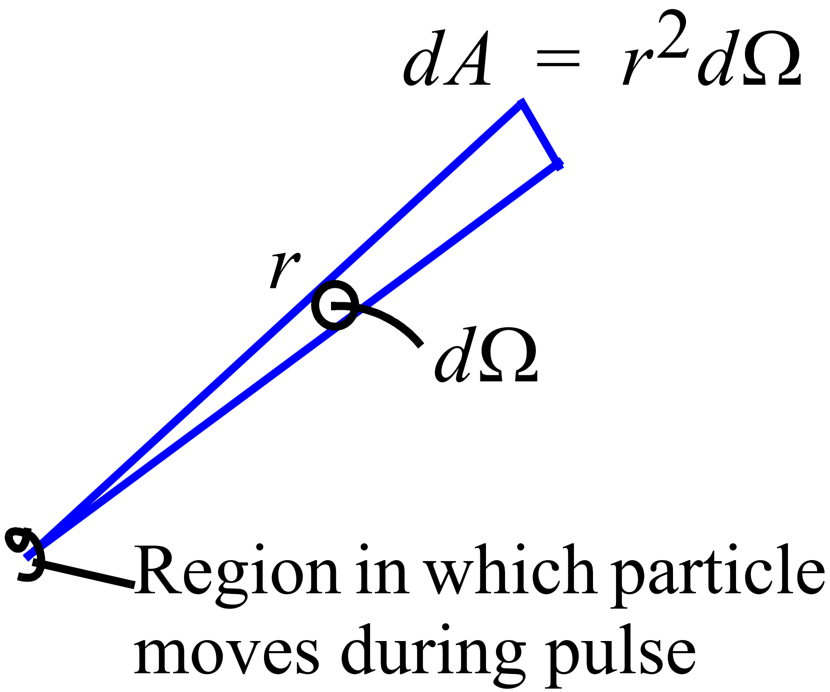
Energy radiated in polarisation mode α into solid angle $d\Omega$, circular frequency range $d\omega$ in time dt

$$= \frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} \times d\Omega \times d\omega \times dt \qquad (25)$$

The total emission coefficient j_ω is defined by

$$j_\omega = \frac{dW_{11}}{d\Omega d\omega dt} + \frac{dW_{22}}{d\Omega d\omega dt} \qquad (26)$$

Variables used to define the emissivity in terms of emitted power.



Consider the surface dA to be located at a distance that is large compared to the distance over which the particle moves when emitting the pulse of radiation. Then $dA = r^2 d\Omega$ and

$$\frac{dW_{\alpha\alpha}}{dA\,d\omega\,dt} = \frac{1}{r^2}\frac{dW_{\alpha\alpha}}{d\Omega\,d\omega\,dt}$$
$$\Rightarrow \frac{dW_{\alpha\alpha}}{d\Omega\,d\omega\,dt} = r^2\frac{dW_{\alpha\alpha}}{dA\,d\omega\,dt} \tag{27}$$

Emissivity corresponding to the $\boldsymbol{e}_\alpha$ component of pulse.

$$\frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} = \frac{c\varepsilon_0 r^2}{\pi T}|E_\alpha(\omega)|^2$$
$$= \frac{c\varepsilon_0 r^2}{\pi T}E_\alpha(\omega)E_\alpha^*(\omega) \text{ (Summation not implied)} \tag{28}$$

3.4 Independence of radius

Note that

$$\frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} \propto r^2|E_\alpha(\omega)|^2 \tag{29}$$

However, we know that for the electric field of a radiating charge

$$E_{\alpha}(t) \propto \frac{1}{r} \rightarrow E_{\alpha}(\omega) \propto \frac{1}{r} \qquad (30)$$

so that

$$\frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} \text{ is independent of } r \qquad (31)$$

3.5 Relationship to the Stokes parameters

We generalise our earlier definition of the Stokes parameters for a plane electromagnetic wave to the following:

$$
\begin{aligned}
I_\omega &= \frac{c\varepsilon_0}{\pi T}[E_1(\omega)E_1^*(\omega) + E_2(\omega)E_2^*(\omega)] \\
Q_\omega &= \frac{c\varepsilon_0}{\pi T}[E_1(\omega)E_1^*(\omega) - E_2(\omega)E_2^*(\omega)] \\
U_\omega &= \frac{c\varepsilon_0}{\pi T}[E_1^*(\omega)E_2(\omega) + E_1(\omega)E_2^*(\omega)] \\
V_\omega &= \frac{1}{i}\frac{c\varepsilon_0}{\pi T}[E_1^*(\omega)E_2(\omega) - E_1(\omega)E_2^*(\omega)]
\end{aligned}
\qquad (32)
$$

The definition of I_ω is equivalent to the definition of specific intensity in the Radiation Field chapter.

Also note the appearance of ***circular*** frequency resulting from the use of the Fourier transform.

To aid the following theoretical development, we define the ***polarisation tensor*** by:

$$I_{\alpha\beta}(\omega) = \frac{1}{2}\begin{bmatrix} I_\omega + Q_\omega & U_\omega - iV_\omega \\ U_\omega + iV_\omega & I_\omega - Q_\omega \end{bmatrix} = \frac{c\varepsilon_0}{\pi T} E_\alpha(\omega) E^*_\beta(\omega) \qquad (33)$$

In the above development, we calculated the emissivities,

$$\frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} = \frac{c\varepsilon_0}{\pi T} r^2 E_\alpha(\omega) E_\alpha^*(\omega) \qquad (34)$$

corresponding to each wave mode. More generally, we define:

$$\frac{dW_{\alpha\beta}}{d\Omega d\omega dt} = \frac{c\varepsilon_0}{\pi T} r^2 E_\alpha(\omega) E_\beta^*(\omega) \qquad (35)$$

and these are the emissivities related to the components of the polarisation tensor $I_{\alpha\beta}$.

In general, therefore, we have

$$
\begin{aligned}
&\frac{dW_{11}}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(I_\omega + Q_\omega) \\
&\frac{dW_{22}}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(I_\omega - Q_\omega) \\
&\frac{dW_{12}}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(U_\omega - iV_\omega) \\
&\frac{dW_{21}}{d\Omega d\omega dt} = \frac{dW_{12}^*}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(U_\omega + iV_\omega)
\end{aligned}
\qquad (36)
$$

Consistent with what we have derived above, the total emissivity is

$$\varepsilon_\omega^I = \frac{dW_{11}}{d\Omega d\omega dt} + \frac{dW_{22}}{d\Omega d\omega dt} \qquad (37)$$

and the emissivity into the Stokes Q is

$$\varepsilon_\omega^Q = \frac{dW_{11}}{d\Omega d\omega dt} - \frac{dW_{22}}{d\Omega d\omega dt} \qquad (38)$$

Also, for Stokes U and V:

$$\varepsilon_\omega^U = \frac{dW_{12}}{d\Omega d\omega dt} + \frac{dW_{12}^*}{d\Omega d\omega dt}$$

$$\varepsilon_\omega^V = i\left(\frac{dW_{12}}{d\Omega d\omega dt} - \frac{dW_{12}^*}{d\Omega d\omega dt}\right) \tag{39}$$

Note that the expression for $dW_{\alpha\beta}/d\Omega d\omega dt$ is independent of radius, r, because of the r^{-1} dependence of the Electric field.

4 Fourier transform of the Lienard-Weichert radiation field

The emissivities for the Stokes parameters depend upon the Fourier transform of

$$rE(t) = \frac{q}{4\pi c\varepsilon_0}\left(\frac{r}{r'}\right)\frac{n' \times [(n' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot n')^3} \tag{40}$$

where

$$t' = t - \frac{r'}{c} \qquad r' = |x - X(t')| \tag{41}$$

The aim of this section is to find the most convenient way of expressing the Fourier transform of $r\boldsymbol{E}(t)$ in terms of the motion of the charge. To begin with we ignore the difference between r and r' since the distance to the field point is large compared to the distance over which the charge moves, i.e. $r/r' \approx 1$.

Transformation to retarded time

The Fourier transform involves an integration wrt t. We transform this to an integral over t' as follows:

$$dt = \frac{\partial t}{\partial t'} dt = (1 - \beta' \cdot \boldsymbol{n}')dt' \qquad (42)$$

To prove that $\frac{\partial t}{\partial t'} = (1 - \beta' \cdot \boldsymbol{n}')$, we proceed as follows:

The relationship between field point time t and source point time (retarded time) is given by:

$$t = t' + \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c} \tag{43}$$

Differentiate wrt t':

$$\frac{\partial t}{\partial t'} = 1 + \frac{1}{c}\frac{\partial}{\partial t'}|\boldsymbol{x} - \boldsymbol{X}(t')| \tag{44}$$

Now,

$$\frac{\partial}{\partial t'}|\boldsymbol{x} - \boldsymbol{X}(t')| = \frac{\partial}{\partial t'}[(x_i - X_i(t'))(x_i - X_i(t'))]^{1/2}$$

$$= \frac{1}{2}\frac{1}{|\boldsymbol{x} - \boldsymbol{X}(t')|} \times 2(x_i - X_i(t')) \times -\frac{\partial}{\partial t'}(X_i(t'))$$

$$\Rightarrow \frac{\partial t}{\partial t'} = 1 - \beta_i(t')n_i' = 1 - \beta' \cdot \boldsymbol{n}' \qquad (45)$$

Hence,

$$\begin{aligned} r\boldsymbol{E}(\omega) &= \frac{q}{4\pi c\varepsilon_0}\int_{-\infty}^{\infty} \frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^3} e^{i\omega t}(1 - \beta' \cdot \boldsymbol{n}')dt' \\ &= \frac{q}{4\pi c\varepsilon_0}\int_{-\infty}^{\infty} \frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^2} e^{i\omega t} dt' \end{aligned} \qquad (46)$$

Integrand in terms of retarded time

The next part is to express

$$e^{i\omega t} = \exp\left[i\omega\left(t' + \frac{r'}{c}\right)\right] \tag{47}$$

in terms of retarded time t'.

Since

$$r' = |\boldsymbol{x} - \boldsymbol{X}(t')| \approx \boldsymbol{x} \quad \text{when} \quad \boldsymbol{x} \gg \boldsymbol{X}(t') \tag{48}$$

then we expand r' to first order in $\boldsymbol{X}$ around $X_i = 0$. Thus,

$$r' = \left|x_j - X_j(t')\right| = r'(0) + \frac{\partial r'}{\partial X_i} X_i$$

$$\frac{\partial r'}{\partial X_i} = \frac{-(x_i - X_i(t'))}{r'} = -\frac{x_i}{r} \text{ at } X_i = 0 \tag{49}$$

$$r' \approx r - \frac{x_i}{r} X_i = r - n_i X_i = r - \boldsymbol{n} \cdot \boldsymbol{X}(t')$$

Note that it is the unit vector $\boldsymbol{n} = \frac{\boldsymbol{r}}{r}$ which enters here, rather than the retarded unit vector $\boldsymbol{n}'$

Hence,

$$\begin{aligned}\exp(i\omega t) &= \exp\left[i\omega\left(t' + \frac{r}{c} - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right)\right] \\ &= \exp\left[i\omega\left(t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right)\right] \times \exp\left[\frac{i\omega r}{c}\right]\end{aligned} \tag{50}$$

The factor $\exp\left[\frac{i\omega r}{c}\right]$ is common to all Fourier transforms $rE_\alpha(\omega)$ and when one multiplies by the complex conjugate to obtain $E_\alpha(\omega)E_\alpha^*(\omega)$ this factor gives unity. This also shows why we expand the argument of the exponential to first order in $\boldsymbol{X}(t')$ since the leading term is eventually unimportant.

Convenient form of integrand

The remaining term to receive attention in the Fourier Transform is

$$\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^2} \qquad (51)$$

We only need to expand the unit vector $\boldsymbol{n}'$ to zeroth order in X_i because the zeroth order term does not disapear. Hence

$$\boldsymbol{n}' \approx \boldsymbol{n} \qquad (52)$$

Therefore,

$$\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^2} \approx \frac{\boldsymbol{n} \times [(\boldsymbol{n} - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n})^2} \qquad (53)$$

It is straightforward (exercise) to show that

$$\frac{d}{dt'}\left[\frac{\boldsymbol{n} \times (\boldsymbol{n} \times \beta')}{1 - \beta' \cdot \boldsymbol{n}'}\right] = \frac{\boldsymbol{n} \times [(\boldsymbol{n} - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n})^2} \tag{54}$$

Hence,

$$rE(\omega) = \frac{q}{4\pi c \varepsilon_0} e^{i\omega r/c} \int_{-\infty}^{\infty} \frac{d}{dt'}\left[\frac{\boldsymbol{n} \times (\boldsymbol{n} \times \beta')}{1 - \beta' \cdot \boldsymbol{n}}\right] \times \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] dt' \tag{55}$$

One can integrate this by parts. First note that

$$\left.\frac{\boldsymbol{n} \times (\boldsymbol{n} \times \beta')}{1 - \beta' \cdot \boldsymbol{n}}\right|_{-\infty}^{\infty} = 0 \tag{56}$$

since we are dealing with a pulse. Second, note that,

$$\frac{d}{dt'} \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] = \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] \times i\omega[1 - \beta' \cdot \boldsymbol{n}] \tag{57}$$

and that the factor of $[1-\beta'\cdot\boldsymbol{n}']$ cancels the remaining one in the denominator. Hence, our final result:

$$r\boldsymbol{E}(\omega) = \frac{-i\omega q}{4\pi c\varepsilon_0}e^{i\omega r/c}\int_{-\infty}^{\infty}\boldsymbol{n}\times(\boldsymbol{n}\times\beta') \times \exp\left[i\omega\left(t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right)\right]dt' \tag{58}$$

In order to calculate the Stokes parameters, one selects a convenient coordinate system ($\boldsymbol{e}_1$ and $\boldsymbol{e}_2$) adapted to the physical situation. The motion of the charge enters through the terms involving $\beta(t')$ and $\boldsymbol{X}(t')$ in the integrand.

Remark on relativistic beaming

The feature associated with radiation from a relativistic particle, namely that the radiation is very strongly peaked in the direction of motion, shows up in the previous form of this integral via the factor $(1 - \beta \cdot \boldsymbol{n}')^{-3}$. This dependence is not evident here. However, when we proceed to evaluate the integral in specific cases, this dependence resurfaces.