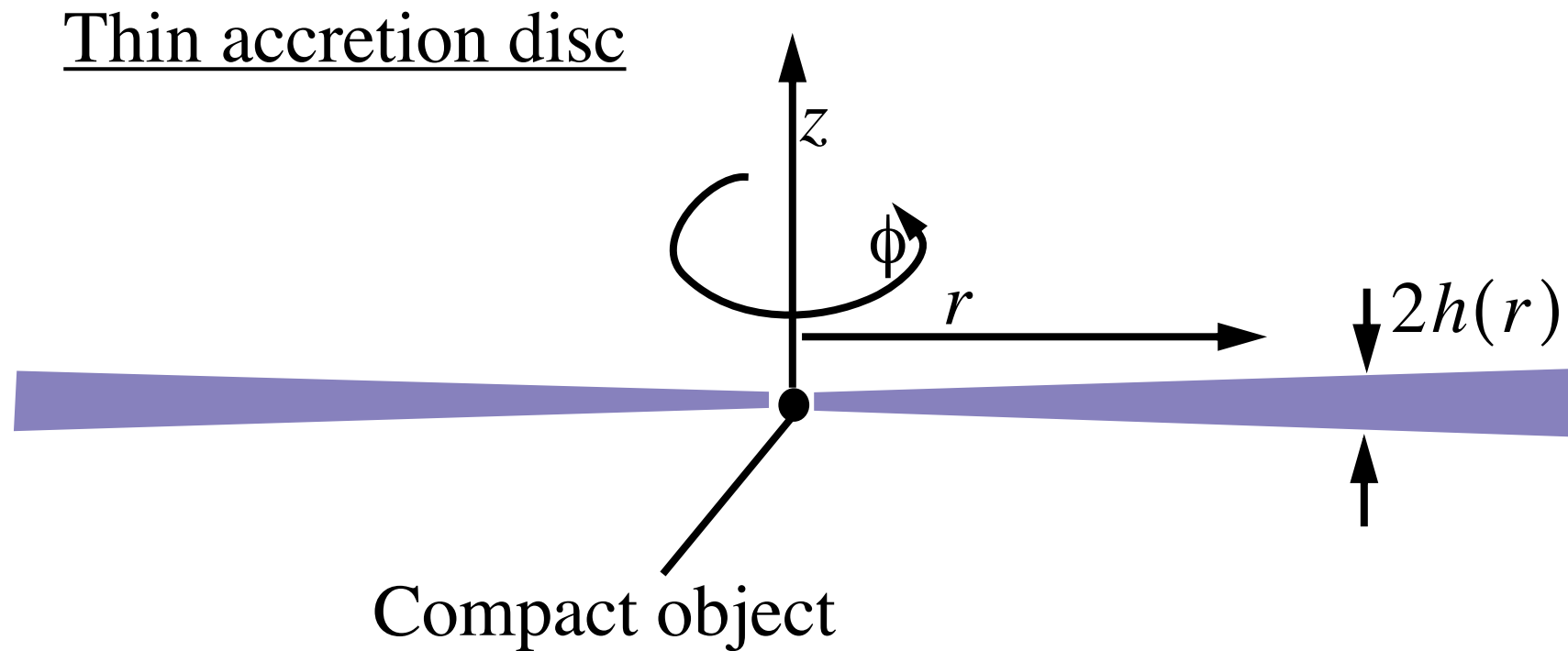# Accretion Disks II

1 The dynamical equations for accretion discs

1.1 Setting up the model

Thin accretion disc

Compact object

The simplest accretion disc model to construct is that of the thin disc. We obtain a simplified set of equations by integrating over the z-dimension and by assuming that the flow is steady in the mean and that it is axisymmetric in the mean. Because of this approximation, we can regard our ensemble average as an azimuthal average, i.e. we average over independent regions around the annulus of an accretion disc.

Assumptions

In the following development of the theory of accretion discs, we assume that:

- The disk is steady in the mean, i.e. time derivatives of the mean flow variables are zero.
- The disk is axisymmetric, i.e. there is no dependence of the mean flow on the azimuthal angle ϕ

- The disk is thin. This means that we can construct useful equations by vertical averaging. The validity of this assumption can be justified *a posteriori*
- The velocity in the disk is dominated by the Keplerian velocity $\tilde{v}_\phi = \sqrt{GM/R}$. In particular $\tilde{v}_r \ll \tilde{v}_\phi$. Again this can be justified ***a posteriori***
- The disk does not have a substantial wind.

Coordinates

- In view of the geometry and the physical assumptions, we use cylindrical polar coordinates, (r, ϕ, z).
- Occasionally use the spherical radius, R.

Mathematical approach

- We develop the accretion disk equations by examining the statistically averaged equations for conservation of mass, momentum and energy in this axisymmetric coordinate system.
- Integrated equations are obtained by integrating the equations over a disk height, i.e. between $z = -h$ and $z = h$.

1.2 Continuity

$$\frac{\partial}{\partial z}(\bar{\rho}\tilde{v}_z) + \frac{1}{r}\frac{\partial}{\partial r}(\bar{\rho}r\tilde{v}_r) = 0 \qquad (1.2\text{-}1)$$

Multiply by $2\pi r$ and integrate over z:

$$2\pi\int_{-h}^{h}\frac{\partial}{\partial z}(\bar{\rho}r\tilde{v}_z)dz + 2\pi\int_{-h}^{h}\frac{\partial}{\partial r}(\bar{\rho}r\tilde{v}_r)dz = 0$$

(1.2-2)

$$\Rightarrow 2\pi r\bar{\rho}\tilde{v}_z\Big|_{-h}^{h} + \frac{d}{dr}\left[2\pi\int_{-h}^{h}(\bar{\rho}r\tilde{v}_r)dz\right] = 0$$

We define the mass accretion rate by:

$$\dot{M}_a = -2\pi\int_{-h}^{h}\bar{\rho}r\tilde{v}_r dz$$

(1.2-3)

We have assumed that there is no wind from the top and bottom surfaces of the disc. Therefore $\tilde{v}_z = 0$ at $z = \pm h$. Hence,

$$\frac{d}{dr}\dot{M}_a = 0 \Rightarrow \dot{M}_a = \text{constant}$$

(1.2-4)

i.e. the accretion rate is constant with radius.

1.3 Momentum

All three components of the momentum equations

$$\frac{\partial}{\partial x_j}(\bar{\rho}\tilde{v}_i\tilde{v}_j) + \frac{\partial}{\partial x_j}\langle \rho v'_i v'_j \rangle = -\frac{\partial \bar{p}}{\partial x_i} - \bar{\rho}\frac{\partial}{\partial x_i}\left(-\frac{GM}{R}\right)$$

$$= -\frac{\partial \bar{p}}{\partial x_i} - \frac{\bar{\rho}GMx_i}{R^3}$$

(1.3-1)

give us essential information.

Since the accretion disc is confined to near $z = 0$ (i.e. $h \ll r$) we treat the gravitational term in the following way. The spherical polar radius

$$R = (r^2 + z^2)^{1/2} \approx r \qquad \text{at} \qquad z \approx 0 \qquad \textit{(1.3-2)}$$

Therefore:

$$\frac{GM}{R^2}\hat{R} = \frac{GM}{R^3}(r, 0, z) \approx \left(\frac{GM}{r^2}, 0, \frac{GM}{r^3}z\right) \qquad \textit{(1.3-3)}$$

In the following we write the hydrodynamical equations in cylindrical polars. This can be accomplished in a number of ways

1. Use the Christoffel symbols for a cylindrical coordinate system to calculate the divergence terms etc.

2. Look up the equations in Landau & Lifshitz

1.4 Radial momentum

The radial momentum balance is given by:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{\rho}\tilde{v}_r^2) + \frac{\partial}{\partial z}(\bar{\rho}\tilde{v}_r\tilde{v}_z) - \frac{1}{r}\bar{\rho}\tilde{v}_\phi^2$$

$$+ \frac{1}{r}\frac{\partial}{\partial r}(r\langle\rho v_r'^2\rangle) + \frac{\partial}{\partial z}\langle\rho v_r' v_z'\rangle \qquad (1.4\text{-}1)$$

$$= -\bar{\rho}\frac{GM}{r^2} - \frac{\partial\bar{p}}{\partial r}$$

We integrate $2\pi r$ times this equation with respect to z and obtain:

$$\frac{d}{dr}\left[2\pi r\int_{-h}^{h}\bar{\rho}\tilde{v}_r^2\,dz\right] + 2\pi r\bar{\rho}\tilde{v}_r\tilde{v}_z\Big|_{-h}^{h} - 2\pi\frac{V_\phi^2}{r}\int_{-h}^{h}\bar{\rho}\,dz$$

$$+ \frac{d}{dr}\left[2\pi r\int_{-h}^{h}\langle\rho v_r'^2\rangle\,dz\right] + 2\pi\langle\rho v_r'v_z'\rangle\Big|_{-h}^{h} \qquad (1.4\text{-}2)$$

$$= -2\pi\frac{GM}{r^2}\int_{-h}^{h}\bar{\rho}\,dz - \frac{d}{dr}\left[2\pi r\int_{-h}^{h}\bar{p}\,dz\right]$$

We neglect all but the blue terms since:

$$\tilde{v}_r, v_r{}' \ll \tilde{v}_\phi \qquad \tilde{v}_z(\pm h) = 0$$

$$\langle \rho v_r{}' v_z{}'\rangle(\pm h) = 0 \qquad p \ll \bar{\rho}\tilde{v}_\phi^2 \qquad \text{(1.4-3)}$$

The last inequality is equivalent to:

$$\frac{\bar{p}}{\bar{\rho}} \ll \tilde{v}_\phi^2 \Rightarrow c_s^2 \ll \tilde{v}_\phi^2 \Rightarrow \frac{\tilde{v}_\phi^2}{c_s^2} \gg 1 \qquad \text{(1.4-4)}$$

that is, the azimuthal speed is highly supersonic. This is justified *a posteriori* below.

Surface density

This important parameter is defined by

$$\Sigma(r) = \int_{-h}^{h} \bar{\rho}\, dz \qquad (1.4\text{-}5)$$

The radial momentum balance equation therefore has the form:

$$2\pi\Sigma(r)\frac{\tilde{v}_\phi^2}{r} = 2\pi\Sigma(r)\frac{GM}{r^2} \qquad (1.4\text{-}6)$$

$$\Rightarrow \tilde{v}_\phi^2 = \frac{GM}{r}$$

Keplerian speed

The Keplerian speed (i.e. the speed of a particle in a circular orbit around the central object) is given by

$$\frac{v_K^2}{r} = \frac{GM}{r^2} \implies v_K^2 = \frac{GM}{r} \qquad (1.4\text{-}7)$$

Hence, the radial momentum equation tells us that

$$\tilde{v}_\phi = v_K \qquad (1.4\text{-}8)$$

Disks in which this equation is valid are called Keplerian.

1.5 Azimuthal momentum

The ϕ-component of the momentum equations is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\bar{\rho}\tilde{v}_r\tilde{v}_\phi) + \frac{\partial}{\partial z}(\bar{\rho}\tilde{v}_\phi\tilde{v}_z)$$

$$+ \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\langle\rho v'_r v'_\phi\rangle] + \frac{\partial}{\partial z}\langle\rho v'_\phi v'_z\rangle = 0 \qquad (1.5\text{-}1)$$

Multiply through by $2\pi r^2$:

$$\frac{\partial}{\partial r}[2\pi r^2\bar{\rho}\tilde{v}_r\tilde{v}_\phi] + \frac{\partial}{\partial z}[2\pi r^2\bar{\rho}\tilde{v}_\phi\tilde{v}_z]$$

$$+ \frac{\partial}{\partial r}[2\pi r^2\langle\rho v'_r v'_\phi\rangle] + \frac{\partial}{\partial z}[2\pi r^2\langle\rho v_\phi' v_z'\rangle] = 0 \qquad (1.5\text{-}2)$$

Integrate over z:

$$\frac{d}{dr}\int_{-h}^{h}(2\pi r\rho\tilde{v}_r)(r\tilde{v}_\phi)dz + 2\pi\rho r^2\tilde{v}_\phi\tilde{v}_z\Big|_{-h}^{h}$$

$$+ \frac{d}{dr}\int_{-h}^{h}2\pi r^2\langle\rho v'_r v'_\phi\rangle dz + 2\pi r^2\langle\rho v'_\phi v'_z\rangle\Big|_{-h}^{h} = 0 \qquad (1.5\text{-}3)$$

Since $\tilde{v}_z = 0$ and $\langle\rho v'_\phi v'_z\rangle(\pm h) = 0$ are both zero at the surface of the disk,

$$\frac{d}{dr}\left[\int_{-h}^{h}(2\pi r\rho\tilde{v}_r)(r\tilde{v}_\phi)dz + \int_{-h}^{h}2\pi r^2\langle\rho v'_r v'_\phi\rangle dz\right] \qquad (1.5\text{-}4)$$
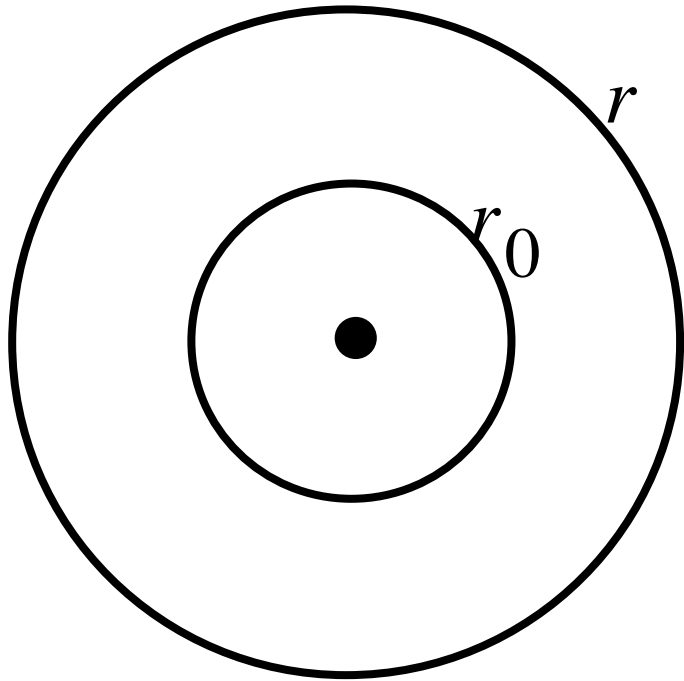
$$= 0$$

The term $r\tilde{v}_\phi$ is the z-component of angular momentum per unit mass.

Write the first term in the brackets as:

$$r\tilde{v}_\phi \times 2\pi\int_{-h}^{h} r\rho\tilde{v}_r dz = -\dot{M}_a(r\tilde{v}_\phi) \qquad (1.5\text{-}5)$$

Integrate the above equation:

$$-\dot{M}_a r\tilde{v}_\phi + 2\pi r^2\int_{-h}^{h}\langle\rho v'_r v'_\phi\rangle dz = \text{Constant} \qquad (1.5\text{-}6)$$

Evaluate the constant using values at the innermost stable orbit.

Also define:

$$G_{r\phi}(r) = \int_{-h}^{h} \langle \rho v'_r v'_\phi \rangle dz \quad \text{(1.5-7)}$$

Hence the azimuthal equation reads:

$$-\dot{M}_a r \tilde{v}_\phi(r) + 2\pi r^2 G_{r\phi}(r)$$

$$= -\dot{M}_a r_0 \tilde{v}_\phi(r_0) + 2\pi r_0^2 G_{r\phi}(r_0) \quad \text{(1.5-8)}$$

Physical meaning of the angular momentum equation

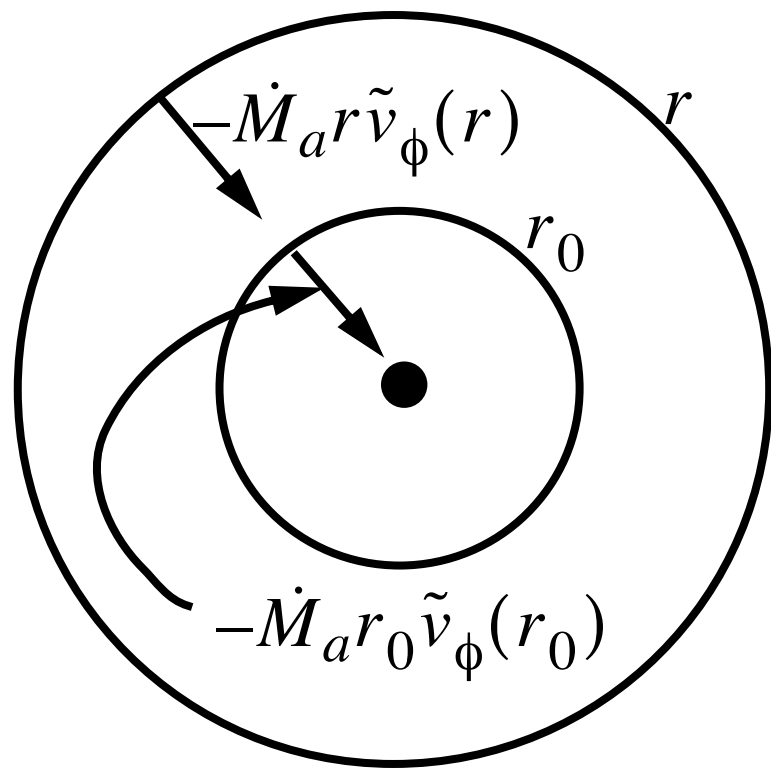Consider $\int_{-h}^{h} (2\pi r \rho \tilde{v}_r)(r\tilde{v}_\phi) dz$

$$r\tilde{v}_\phi = \begin{array}{l} z \text{ component of angular} \\ \text{momentum per unit mass} \end{array} \quad (1.5\text{-}9)$$

$$(\bar{\rho}\tilde{v}_r) \times r\tilde{v}_\phi = \begin{array}{l} \text{Flux of } z \text{ component of angular} \\ \text{momentum in } r \text{ direction} \end{array} \quad (1.5\text{-}10)$$

$$2\pi r dz = \text{Element of area} \quad (1.5\text{-}11)$$

Therefore:

$$\int_{-h}^{h} (2\pi r \rho \tilde{v}_r)(r\tilde{v}_\phi) dz = \begin{array}{l} \text{Flux of angular momentum} \\ \text{in } r \text{ direction} \end{array} \quad (1.5\text{-}12)$$

As we have seen this flux is negative and we have put it equal to $-\dot{M}_a r \tilde{v}_\phi$, i.e. accretion implies that there is a flux of angular momentum inwards.

Also, since $\dot{M}_a$ is constant, the flux of angular momentum is larger at larger r because $r\tilde{v}_\phi(r) = (GMr)^{1/2}$

This means that as a result of the accretion there is more angular momentum transported by accretion into the annulus between r and r_0 than is transported out at the smaller radius $r = r_0$.

Now consider:

$$2\pi r^2 \int_{-h}^{h} \langle \rho v'_r v'_\phi \rangle \, dz \qquad (1.5\text{-}13)$$

$$\langle \rho v'_r v'_\phi \rangle = \begin{array}{l} \text{Flux of turbulent } \phi \text{ momentum density} \\ \text{in } r \text{ direction} \end{array} \qquad (1.5\text{-}14)$$

$$r \langle \rho v'_r v'_\phi \rangle = \begin{array}{l} \text{Flux of } z \text{ component of turbulent} \\ \text{angular momentum in } r \text{ direction} \end{array} \qquad (1.5\text{-}15)$$

Therefore

$$2\pi r \int_{-h}^{h} r \langle \rho v'_r v'_\phi \rangle \, dz = \begin{array}{l} \text{Turbulent flux of angular} \\ \text{momentum in } r \text{ direction} \end{array} \qquad (1.5\text{-}16)$$

It is this flux that balances the build up of angular momentum between r and r_0.

The equation:

$$- \dot{M}_a r \tilde{v}_\phi(r) + 2\pi r^2 G_{r\phi}(r)$$

$$= - \dot{M}_a r_0 \tilde{v}_\phi(r_0) + 2\pi r_0^2 G_{r\phi}(r_0) \qquad \text{(1.5-17)}$$

tells us that

Flux of angular momentum through r = Flux of angular momentum through r_0 (1.5-18)

That is angular momentum is conserved in between r and r_0.

This is correct since there are no torques acting on the disk material.

Note also that this analysis applies to any annulus, not just one involving the innermost stable orbit.

We can now solve for $G_{r\phi} = \int_{-h}^{h} \langle \rho v'_r v'_\phi \rangle dz$:

$$G_{r\phi}(r) = \left(\frac{r_0}{r}\right)^2 G_{r\phi}(r_0) + \frac{\dot{M}_a \tilde{v}_\phi}{2\pi r}\left[1 - \frac{r_0 \tilde{v}_\phi(r_0)}{r \tilde{v}_\phi(r)}\right]$$

$$= \left(\frac{r_0}{r}\right)^2 G_{r\phi}(r_0) + \frac{\dot{M}_a \tilde{v}_\phi}{2\pi r}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]$$

(1.5-19)

The effect of the inner boundary condition decreases quite rapidly with r so that we often neglect it and take:

$$G_{r\phi}(r) = \frac{\dot{M}_a \tilde{v}_\phi}{2\pi r}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]$$

(1.5-20)

1.6 Vertical equilibrium

The momentum equation in the z—direction reads:

$$\frac{1}{r}\frac{\partial}{\partial r}[r\langle \rho v_r' v_z'\rangle + \rho\tilde{v}_r\tilde{v}_z] + \frac{\partial}{\partial z}[\bar{\rho}v_z^2]$$

$$= -\frac{\partial}{\partial z}[\bar{p} + \langle \rho v_z'^2\rangle] - \frac{GM}{r^2}z$$

(1.6-1)

We assume:

1. There is no wind ($\tilde{v}_z = 0$).

2. The turbulent stresses are much less than the pressure ($\langle \rho v'^2 \rangle \ll \bar{p}$). This means that we can neglect the terms involving $\langle \rho v_r' v_z' \rangle$ and $\langle \rho v_z'^2 \rangle$.

Thus the equation for vertical equilibrium reduces to:

$$\frac{\partial}{\partial z}\bar{p} = -\bar{\rho}\frac{GMz}{r^3} \qquad \textit{(1.6-2)}$$

Isothermal disk

In order to get a quantitative feel for the implications of this equation, let us assume that the disk is isothermal.

We put

$$\bar{p} = \bar{\rho}\frac{k\tilde{T}}{\mu m_p}$$

(1.6-3)

With $\tilde{T} = $ constant

$$\frac{k\tilde{T}}{\mu m_p}\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\bar{\rho} = -\frac{GM}{r^3}z$$

(1.6-4)

$$\Rightarrow \frac{\bar{\rho}}{\bar{\rho}_c} = \exp\left[-\frac{\mu GM m_p}{2k\tilde{T}r^3}z^2\right]$$

This defines the disk scale height

$$h_s^2 = \frac{2kTr^3}{\mu GMm_p} \Rightarrow \left(\frac{h_s}{r}\right)^2 = 2\left(\frac{kT}{\mu m_p}\right)\left(\frac{GM}{r}\right)^{-1}$$
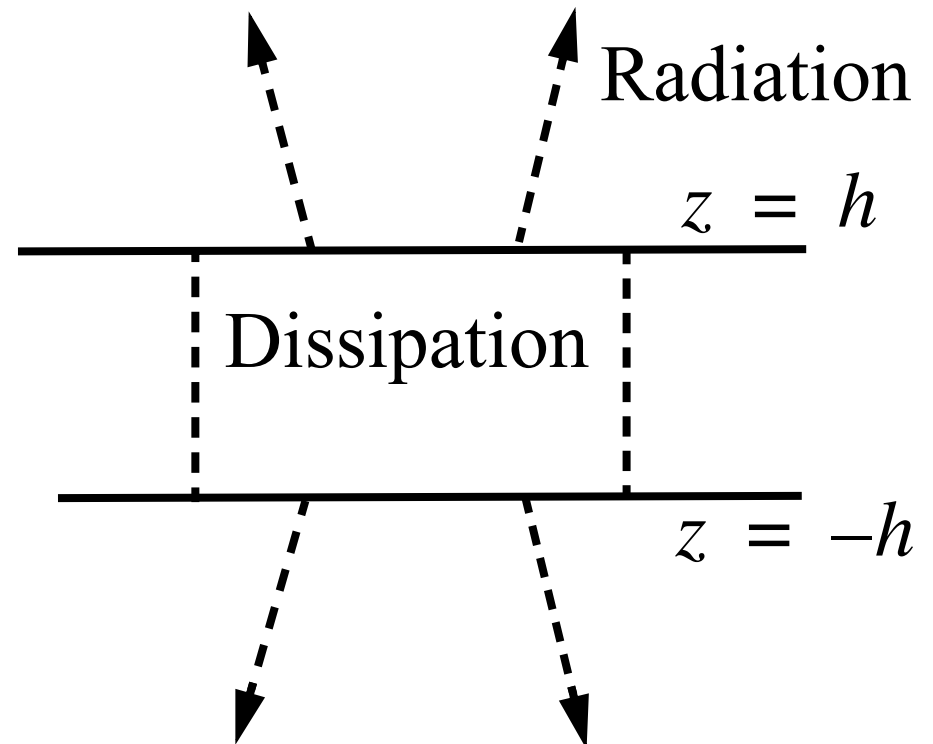
$$= 2\frac{a_s^2}{v_K^2} = \frac{2}{M_K^2}$$

(1.6-5)

Hence the condition that the disc be thin is equivalent to the condition that the Mach number of the Keplerian flow be supersonic.

1.7 The energy equation

In the following we take a section of an accretion disk and calculate the radiation emitted from the surface as a result of the dissipation within the disk.

Radiation

$z = h$

Dissipation

$z = -h$

Rate of production of turbulent energy

Knowing the Reynolds stress from the angular momentum equation, we can evaluate the local rate of production of turbulent energy. This is

$$-\langle \rho v_i' v_j' \rangle \tilde{s}_{ij} \approx -2\langle \rho v_r' v_\phi' \rangle \tilde{s}_{r\phi} \qquad (1.7\text{-}1)$$

where the $r\phi$ component of shear is given by:

$$\tilde{s}_{r\phi} = \frac{1}{2}r\Omega' \qquad (1.7\text{-}2)$$

The angular velocity is given by the Keplerian value

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2} \qquad (1.7\text{-}3)$$

The derivative of this quantity is:

$$\Omega' = -\frac{3}{2}\frac{(GM)^{1/2}}{r^{5/2}} = -\frac{3}{2}\frac{\Omega}{r} \qquad (1.7\text{-}4)$$

The production rate per unit volume of turbulent energy is therefore

$$\Lambda(r, z) = -\langle \rho v_i' v_j'\rangle \tilde{s}_{ij} \approx -\langle \rho v_r' v_\phi'\rangle r\Omega'$$

$$= \frac{3}{2}\langle \rho v_r' v_\phi'\rangle \Omega \qquad (1.7\text{-}5)$$

The production rate of turbulent energy **per unit area** of the disk is

$$\Lambda(r) = \int_{-h}^{h} -\langle \rho v_r{}' v_\phi{}' \rangle r\Omega' \, dz = \frac{3\Omega}{2} \int_{-h}^{h} \langle \rho v_r{}' v_\phi{}' \rangle \, dz$$

(1.7-6)

$$= \frac{3\Omega}{2} G_{r\phi}(r) = \frac{3\dot{M}v_K\Omega}{4\pi r}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]$$

where we have used the result from the analysis of the angular momentum equation:

$$G_{r\phi}(r) = \frac{\dot{M}_a \tilde{v}_\phi}{2\pi r}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]$$

(1.7-7)

Now

$$v_K \Omega = \left(\frac{GM}{r}\right)^{1/2}\left(\frac{GM}{r^3}\right)^{1/2} = \frac{GM}{r^2} \qquad (1.7\text{-}8)$$

so that

$$\Lambda(r) = \int_{-h}^{h} -2\langle \rho v_r' v_\phi' \rangle \tilde{s}_{r\phi} dz$$

$$= \frac{3GM\dot{M}_a}{4\pi r^3}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right] \qquad (1.7\text{-}9)$$

Radiative flux

We assume that the production of turbulence per unit area is equal to the dissipation into heat per unit area and that this heat is radiated in a quasi steady state away from the surface of the disk. The luminosity of the disk emitted between radii r_1 and r_2 is

$$L_{\text{disc}}(r_1, r_2) = \frac{3GM\dot{M}_a}{4\pi}\int_{r_1}^{r_2}\frac{1}{r^3}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]2\pi r\, dr$$

$$= \frac{3GM\dot{M}_a}{2r_0}\int_{r_1}^{r_2}\left(\frac{r_0}{r}\right)^2\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]d\left(\frac{r}{r_0}\right) \qquad (1.7\text{-}10)$$

$$= \frac{3GM\dot{M}_a}{2}\left\{\frac{1}{r_1}\left[1 - \frac{2}{3}\left(\frac{r_0}{r_1}\right)^{1/2}\right] - \frac{1}{r_2}\left[1 - \frac{2}{3}\left(\frac{r_0}{r_2}\right)^{1/2}\right]\right\}$$

The total luminosity emitted by the disc is obtained by integrating between r_0 and ∞:

$$L = L_{\text{disc}}(r_0, \infty) = \frac{GM\dot{M}_a}{2r_0} \qquad (1.7\text{-}11)$$

Relationship to overall energetics

Consider the energy of a parcel of gas with a mass, m, at radius r. This is:

$$E_b = \frac{1}{2}mv_K^2 - \frac{GMm}{r} = m\left(\frac{1}{2}\frac{GM}{r} - \frac{GM}{r}\right)$$

$$= -\frac{1}{2}\frac{GMm}{r}$$

(1.7-12)

This gas has been accreted, essentially from $r = \infty$ so that this represents the energy that has been lost by this parcel of gas during the time that it has spiralled in from $r = \infty$ to r.

By the time the gas has reached the innermost stable orbit, the energy lost per unit mass is:

$$\frac{GM}{2r_0} \hspace{5cm} (1.7\text{-}13)$$

The mass accreted per unit time is $\dot{M}_a$. Hence, the total power associated with the accretion is

$$\dot{M}_a \times \frac{GM}{2r_0} = \frac{GM\dot{M}_a}{2r_0} \hspace{3cm} (1.7\text{-}14)$$

Hence the total luminosity of the disk is the power associated with the energy lost per unit mass at radius r_0 times the mass flux.

The gravitational radius

With black holes or neutron stars in mind, we express the radius in units of the gravitational radius

$$r_g = \frac{GM}{c^2} \qquad (1.7\text{-}15)$$

In terms of the gravitational radius, the luminosity is

$$L = \frac{GM\dot{M}_a}{2r_0} = \frac{GM\dot{M}_a}{2r_g(r_0/r_g)} = \frac{\dot{M}_a c^2}{2(r_0/r_g)} \qquad (1.7\text{-}16)$$

For a spherical black hole, the innermost stable orbit is at 6 gravitational radii. If our Newtonian treatment were to be adequate at such radii

$$L \approx 0.083 \dot{M}_a c^2 \qquad (1.7\text{-}17)$$

i.e. approximately 8% of the infalling rest mass energy is converted into radiation. The exact answer from the general relativistic treatment is $L \approx 0.057 \dot{M}_a c^2$. That is, 5.7% of the mass energy is converted into luminosity. For a rotating black hole the figure goes up to 42%. Typically, for order of magnitude purposes, we assume that black holes are 10% efficient.

Neutron stars are actually slightly more efficient than spherical black holes because the kinetic energy of the infalling material is converted into radiation at a shock on the surface.

2 Accretion disk temperature

2.1 Use of Stefan's law for an optically thick disk

If we assume that the disc is optically thick (true in a large number of cases) then the dissipation per unit area appearing in radiation from *each side* of the disc is

$$\frac{1}{2}\Lambda(r) = \frac{3GM\dot{M}_a}{8\pi r^3}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right] \qquad (2.1\text{-}1)$$

The temperature of the disk surface is given by

$$\sigma T^4 = \frac{3c^6}{4\pi G^2 M^2}\frac{\dot{M}_a}{}\left(\frac{r}{r_g}\right)^{-3}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right] \qquad (2.1\text{-}2)$$

Hence,

$$T = \left(\frac{3c^6}{4\pi\sigma G^2}\right)^{1/4}\left(\frac{\dot{M}_a^{1/4}}{M^{1/2}}\right)\left(\frac{r}{r_g}\right)^{-3/4}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]^{1/4} \qquad (2.1\text{-}3)$$

The constant

$$\left(\frac{3c^6}{4\pi\sigma G^2}\right)^{1/4} = 2.88\times10^{19} \text{ SI units} \qquad (2.1\text{-}4)$$

Typical parameters for a galactic mass black hole are $\dot{M}_a = 10^{14}$ Kg/s and $M = 3M_O$ giving

$$L = 7.5 \times 10^{29} \text{W}$$

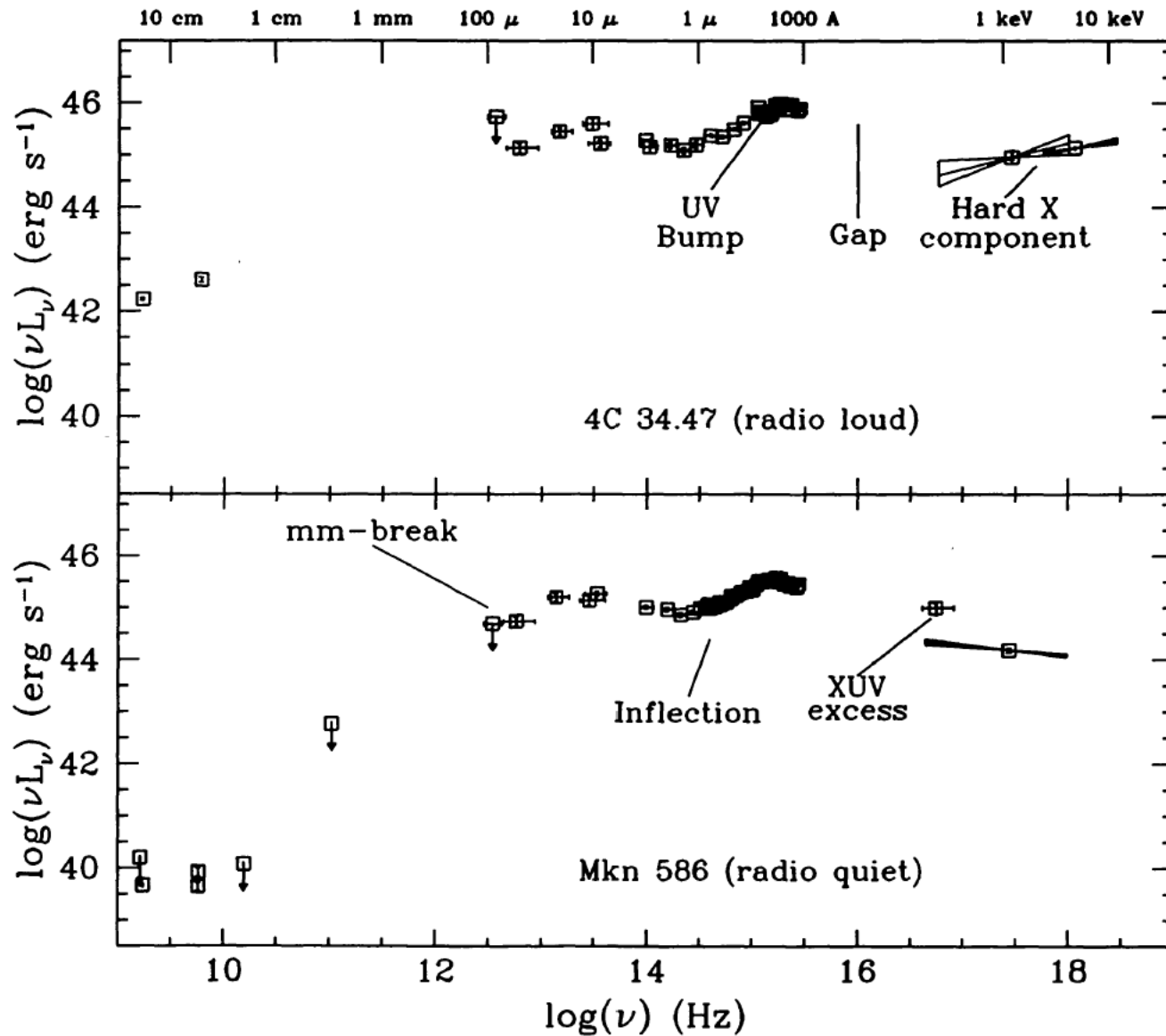$$T = 3.7 \times 10^7 \left(\frac{r}{r_g}\right)^{-3/4} \left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]^{1/4} {}^{\circ}\text{K} \qquad (2.1\text{-}5)$$

Typical parameters for an extragalactic supermassive black hole in an AGN are $\dot{M}_a = 0.1 M_O$/yr and $M = 10^8 M_O$ giving:

$$L \approx 4.7 \times 10^{37} \text{ W}$$

$$T \approx 5.7 \times 10^5 \left(\frac{r}{r_g}\right)^{-3/4} \left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]^{1/4} \text{ °K} \qquad (2.1\text{-}6)$$

The emission for a galactic mass black hole therefore peaks in the X-ray; that from a supermassive black hole peaks in the UV.

This the reason proposed for the UV bumps shown in the spectra of AGN at the left.

3 The Eddington Luminosity

3.1 Derivation for spherically symmetric accretion

The luminosity of an accreting object cannot increase indefinitely. There is a fundamental limit, known as the Eddington limit, which limits accretion by the radiation force on the accreting matter.
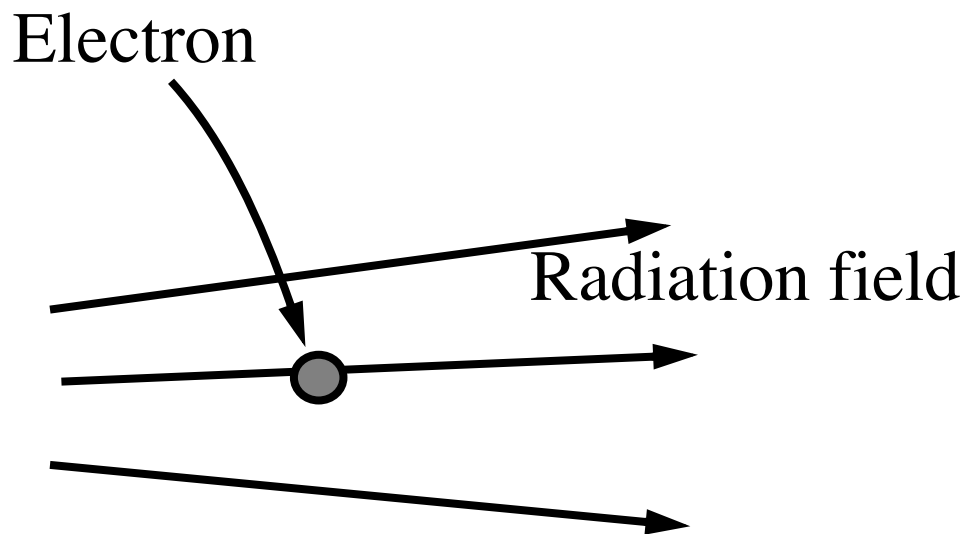
An electron in a radiation field feels a force proportional to the momentum flux density, P_{rad}, of the radiation field. This is given by

$$F = \sigma_T P_{rad} \qquad (3.1\text{-}1)$$

(Intuitively, one can think of this expression as Force = Pressure × Area.)

σ_T is the Thomson cross-section given by

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2 = \frac{8\pi r_0^2}{3} = 6.65\times10^{-29} \ \text{m}^2 \quad (3.1\text{-}2)$$

where the electron radius,

$$r_0 = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = 2.818\times10^{-15} \ \text{m} \qquad (3.1\text{-}3)$$
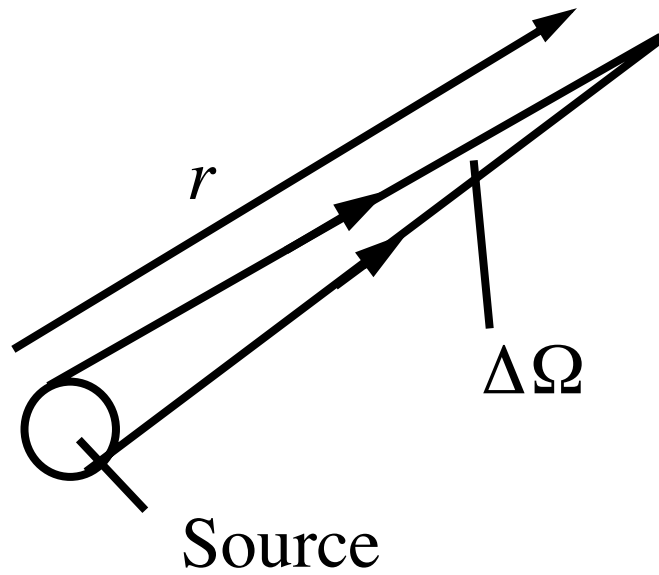
Calculation of momentum flux density

The momentum flux density per unit frequency is given by:

$$P_\nu = \frac{1}{c} \int_{source} I_\nu \cos^2\theta \, d\Omega \qquad (3.1\text{-}4)$$

and the flux density is

$$F_\nu = \int_{source} I_\nu \cos\theta \, d\Omega \qquad (3.1\text{-}5)$$

For distances large compared to the dimensions of the source

$$F_\nu \approx I_\nu \Delta\Omega$$

$$P_\nu \approx \frac{I_\nu}{c}\Delta\Omega = \frac{F_\nu}{c} \qquad (3.1\text{-}6)$$

For an isotropic emitter the total luminosity is spread over the radius of a sphere at the distance r. Hence,

$$F_\nu \times 4\pi r^2 = L_\nu \Rightarrow F_\nu = \frac{L_\nu}{4\pi r^2}$$

$$\Rightarrow P_\nu = \frac{L_\nu}{4\pi r^2 c} \Rightarrow P_{rad} = \frac{L}{4\pi r^2 c}$$

(3.1-7)

where P_{rad} and L both involve integrations over the frequency, ν. Hence, the force on an electron is

$$F = \frac{L\sigma_T}{4\pi r^2 c}$$

(3.1-8)

Now consider a hydrogen plasma consisting of electrons and protons in the gravitational field of an object of mass M. Consider the nett outward force exerted on each electron-proton pair.

The radiation force is primarily exerted on the electron but the gravitational force is primarily exerted on the proton. However, the two cannot move apart since this would result in a large charge separation. The nett force on the electron-proton pair is:

$$F_{\text{nett}} = \frac{L\sigma_T}{4\pi r^2 c} - \frac{GMm_p}{r^2} \qquad (3.1\text{-}9)$$

If the above nett force is greater than zero, accretion cannot occur. Thus, for accretion,

$$\frac{L\sigma_T}{4\pi r^2 c} - \frac{GMm_p}{r^2} < 0$$

$$\Rightarrow L < \frac{4\pi GMm_p c}{\sigma_T} = 1.3 \times 10^{31} \left(\frac{M}{\text{Solar mass}}\right) \text{ W}$$

(3.1-10)

The parameter

$$L_{\text{edd}} = \frac{4\pi GMcm_p}{\sigma_T}$$

(3.1-11)

is known as the Eddington luminosity. Although this limit has been derived here for the case of spherical accretion, this limit is an important benchmark in all accretion scenarios. This limit was originally derived by Eddington in the context of stars.

For compact objects of order a solar mass in size, the Eddington luminosity is of order 10^{31} W. For black holes of order 10^{9} solar masses, the Eddington luminosity is of order 10^{40} W. It is therefore not surprising that these luminosities represent the upper limits of what is normally observed in these environments.

3.2 The Eddington accretion rate

Consider now the case where the luminosity of the central source is actually derived from accretion. As we have seen for an accretion disc, we can represent the total luminosity in the form:

$$L = \alpha \dot{M}_a c^2 \qquad (3.2\text{-}1)$$

where $\alpha \sim 0.1$ for a black hole. Hence, in order to satisfy the Eddington constraint:

$$\alpha \dot{M}_a c^2 < \frac{4\pi GM m_p c}{\sigma_T}$$

$$\Rightarrow \dot{M}_a < \alpha^{-1} \frac{4\pi GM m_p}{c\sigma_T} \qquad (3.2\text{-}2)$$

The parameter

$$\dot{M}_{edd} = \frac{4\pi GMm_p}{c\sigma_T} = 1.4\times10^{14}\left(\frac{M}{M_{solar}}\right) \text{ kg s}^{-1}$$

$$= 2.2\times10^{-9}\left(\frac{M}{M_{solar}}\right) M_{solar} \text{ yr}^{-1}$$

(3.2-3)

Thus, 10^{14} kg s^{-1} is the maximum sort of accretion rate that one expects in a solar mass sized object. For an object with a size of order 10^9 solar masses, one expects accretion rates up to about a solar mass per year.

4 The viscosity prescription – the α parameter

4.1 General approach

A large number of the relationships we have derived have been without a specific prescription for the Reynolds stress. All of our results have been expressed in terms of the accretion rate. This is typical of what we often do when modelling turbulent flows, e.g. jets. However, to derive other information e.g. the inflow velocity $\tilde{V}_r$ we need to be more prescriptive about the turbulent model.

As we indicated earlier the closure relations for turbulence are difficult to obtain. Shakura & Sunyaev in their classic paper on accretion discs lumped all of the unknowns into one simple equation:

$$\langle \rho v_r{}' v_\phi{}' \rangle \approx \alpha P \qquad (4.1\text{-}1)$$

Hence

$$\alpha \sim \frac{\langle \rho v'^2 \rangle}{P} \sim \frac{v'^2}{c_s^2} \qquad (4.1\text{-}2)$$

Normally, in a turbulent flow, the turbulent velocity is less than the sound speed. There are 2 reasons for this:

1. In a turbulent flow sound waves are emitted and the rate of emission goes up as the 8th power of the Mach number of the

turbulence. This means that the emission of these waves would be a very effective dampener of supersonic turbulence.

2.Supersonic turbulence would very quickly form shocks.

Therefore, we usually assume that

$$\alpha < 1 \tag{4.1-3}$$

and frequently we find that

$$\alpha \ll 1 \tag{4.1-4}$$

4.2 Use of the α–parameter to obtain the inflow velocity

We already have:

$$2\pi r^2 \int_{-\infty}^{\infty} \langle \rho v_r' v_\phi' \rangle \, dz = r\dot{M}_a v_K \left[1 - \left(\frac{r_0}{r} \right)^{1/2} \right] \qquad (4.2\text{-}1)$$

and using the α prescription

$$2\pi r^2 \alpha \int_{-h}^{h} \bar{\rho} \left(\frac{\bar{p}}{\bar{\rho}} \right) dz = r\dot{M}_a v_K \left[1 - \left(\frac{r_0}{r} \right)^{1/2} \right] \qquad (4.2\text{-}2)$$

That is,

$$2\pi r^2 \alpha \int_{-h}^{h} \bar{\rho} \left(\frac{k\tilde{T}}{\mu m_p} \right) dz = r\dot{M}_a v_K \left[1 - \left(\frac{r_0}{r} \right)^{1/2} \right] \qquad (4.2\text{-}3)$$

Assuming isothermality:

$$2\pi r^2 \alpha \left(\frac{k\tilde{T}}{\mu m_p}\right) \Sigma = r\dot{M}_a v_K \left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]$$

$$\Rightarrow 2\pi r \Sigma = \alpha^{-1} \left(\frac{kT}{\mu m_p}\right)^{-1} \dot{M}_a v_K \left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]$$

(4.2-4)

Now the mass accretion rate is

$$\dot{M}_a = -2\pi r \int_{-h}^{h} \rho \tilde{v}_r dz = -\Sigma 2\pi r \tilde{v}_r$$

(4.2-5)

$$\Rightarrow \tilde{v}_r = -\frac{\dot{M}_a}{2\pi r \Sigma}$$

Therefore:

$$\left|\tilde{v}_r\right| = \alpha\left(\frac{kT}{\mu m_p}\right)v_K^{-1}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]^{-1}$$

$$= \alpha\frac{a_s^2}{v_K}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]^{-1} \qquad \textit{(4.2-6)}$$

$$= \alpha a_s M_K^{-1}\left[1 - \left(\frac{r_0}{r}\right)^{1/2}\right]^{-1}$$

Hence, for $\alpha < 1$ the inflow velocity is much less than the sound speed.

5 The effect of magnetic fields

The influence of magnetic fields on accretion disc dynamics is very much a current research topic. Let me just note the equation for the angular momentum of the disc (the ϕ—component of the momentum equations) which reads

$$\frac{d}{dr}\left[\dot{M}_a r \tilde{v}_\phi + \frac{1}{2}r^2 \langle B_r' B_\phi' \rangle - \int_{-\infty}^{\infty} 2\pi r^2 \langle \bar{\rho} v'_r v'_\phi \rangle dz\right] \quad \text{(5.0-1)}$$

$$= 0$$

The term $\langle B_r' B_\phi' \rangle$ are related to the flux of the z—component of angular momentum in the radial direction by the magnetic field.

It is important to consider the magnetic field since it is impossible to make an unmagnetised Keplerian disc turbulent. However, it has been shown by Balbus and Hawley that a weak magnetic field produces instability. Hence, the present focus on using magnetic fields to transport angular momentum (cf. the case for winds). The B_r and B_ϕ components which are responsible for this are supposed to be generated by the Balbus-Hawley instabilities. If this is the case then the dissipation in the disc will be through reconnection of magnetic fields. Much of the analysis that we have carried out here for unmagnetised disks carries through for magnetised disks.