Accretion Disks I

References:

Accretion Power in Astrophysics, J. Frank, A. King and D. Raine.

High Energy Astrophysics, Vol. 2, M.S. Longair, Cambridge University Press

Active Galactic Nuclei, J.H. Krolik, Princeton Series in Astrophysics

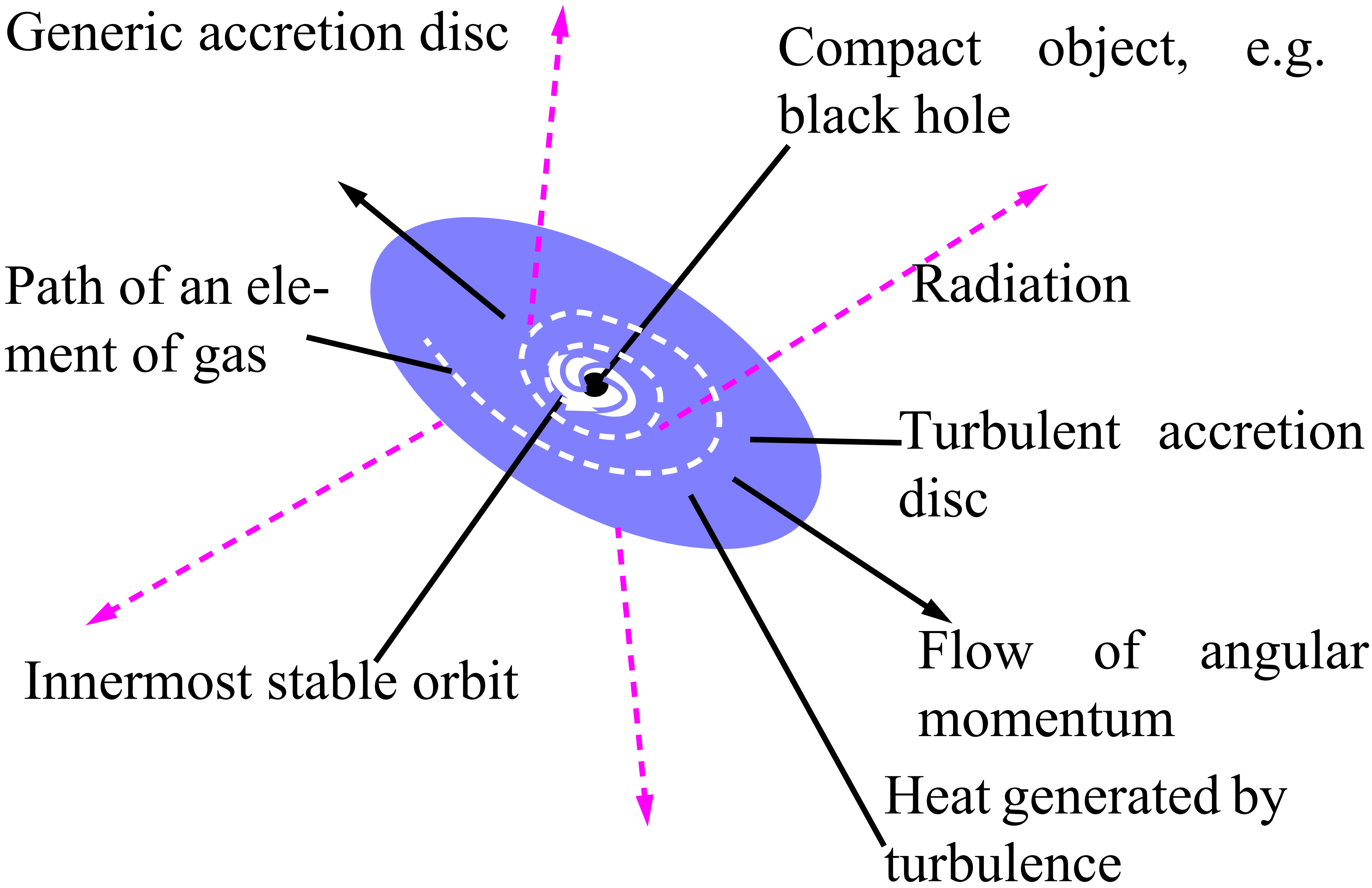
1 Overview

Accretion disks are important in a number of areas of astrophysics

- Accretion on stellar mass size compact objects (white dwarfs, neutron stars, black holes).
- Accretion onto newly forming protostars.
- Accretion onto supermassive ($\sim 10^{7-9}$ solar masses) black holes in Active Galactic Nuclei.

The physics of accretion in these different environments is generic and much can be understood from a general theory. In this set of lectures therefore, we are concerned with the generic physics of accretion discs. This involves:

1. Some general characteristics of turbulent flow. Turbulence is important in accretion discs as a source of “viscosity” which drives the accretion.
2. The derivation of equations of turbulent flow which are specific to accretion discs
3. The radiation properties of accretion discs.
4. The importance of magnetic fields to understanding the fundamental properties of accretion discs. We shall begin by deriving equations which relate to unmagnetised discs. However, we shall see that we cannot do without magnetic fields. This effectively brings us to current research on accretion discs.


Generic accretion disc
Compact object, e.g. black hole
Path of an element of gas
Radiation
Turbulent accretion disc
Flow of angular momentum
Innermost stable orbit
Heat generated by turbulence

2 The equations of fluid dynamics with viscosity

2.1 General equations

In accretion discs, dissipation is important and only avenue of dissipation available in an unmagnetised fluid is viscosity. We return to the general equations of fluid dynamics. Note that in the following we do not include the dynamical effects of magnetic fields. This suffices to introduce us to the general concepts of accretion discs. However, in the long run, we cannot do without magnetic fields. These are the ultimate source of the dynamic stresses which lead to accretion.

Continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \qquad (2.1\text{-}1)$$

Momentum:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p_{ij}}{\partial x_j} - \rho\frac{\partial\phi_G}{\partial x_i}$$

$$= -\frac{\partial p}{\partial x_i} + \frac{\partial\sigma_{ij}}{\partial x_j} - \rho\frac{\partial\phi_G}{\partial x_i} \qquad (2.1\text{-}2)$$

The quantity σ_{ij} is the ***viscous tensor*** which is expressed in the form

$$\sigma_{ij} = 2\eta S_{ij} + \zeta v_{k,k}\delta_{ij}$$

$$s_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i} - \frac{2}{3}v_{k,k}\delta_{ij}\right) = \text{Shear tensor} \qquad (2.1\text{-}3)$$

The parameters η and ζ are the coefficients of shear and bulk dynamic ***molecular viscosity***. The corresponding coefficients of kinematic viscosity are

$$\mu = \frac{\eta}{\rho} \qquad \chi = \frac{\zeta}{\rho}, \qquad (2.1\text{-}4)$$

and to order of magnitude

$$(\mu, \chi) \sim \text{Mean free path} \times \text{Thermal speed} \qquad (2.1\text{-}5)$$

Note that the dimensions of kinematic viscosity are length × velocity.

In classical applications of viscous flow, e.g. fluid flow in a pipe, the detailed form of the viscous tensor is important. However, for our application to turbulent flow, it is not important.

2.2 Reynolds number

The Reynolds number is defined as:

$$R = \frac{VL}{\nu} \qquad (2.2\text{-}1)$$

and in most astrophysical flows is large.

2.3 Dissipation due to viscosity

The heat generated by a viscous fluid is described by:

$$\rho kT\frac{ds}{dt} = v_{i,j}\sigma_{ij} - q_{j,j} \tag{2.3-1}$$

$$q_i = \text{Heat flux} \tag{2.3-2}$$

Since the rate of change of the entropy per unit mass can be expressed in the form:

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon + p)}{\rho}\frac{d\rho}{dt} = \frac{d\varepsilon}{dt} + (\varepsilon + p)\frac{\partial v_i}{\partial x_i} \tag{2.3-3}$$

then we have the equation for the change of internal energy as a result of viscous dissipation:

$$\frac{d\varepsilon}{dt} + (\varepsilon + p)\frac{\partial v_i}{\partial x_i} = \sigma_{ij} v_{i,j} - \frac{\partial q_i}{\partial x_i} \qquad (2.3\text{-}4)$$

This equations tells us that there is dissipation resulting from viscosity and loss of energy from a comoving volume resulting from the heat flux.

3 Equations for statistically averaged turbulent flow

3.1 Definitions of mean dynamic variables

Means of density and velocity

The following formalism is suitable for treating all turbulent flows. We envisage a turbulent flow as consisting of a mean plus a fluctuating component so that we express the density and velocity in the form:

$$\rho = \bar{\rho} + \rho' \qquad v_i = \tilde{v}_i + v_i' \tag{3.1-1}$$

and the means of the fluctuating components are defined by

$$\langle \rho' \rangle = 0 \qquad \langle \rho v_i' \rangle = 0 \tag{3.1-2}$$

“Mean” in this case refers to either an ensemble mean (i.e. we envisage an infinite population of accretion discs, or a time averaged mean in which we average over a large enough time scale that the averaged flow has a smooth character).

Note that the velocity is defined by a mass-weighted mean. This is similar to the way in which we define the bulk velocity V_i of a fluid – by defining it in terms of the velocity of the frame in which the total *momentum* is zero.

Means of other quantities

Other variables are also averaged and whether they are mass-weighted or not depends upon their dimensions and the way in which they appear in the hydrodynamic equations:

$$p = \bar{p} + p' \qquad \sigma_{ij} = \bar{\sigma}_{ij} + \sigma_{ij}'$$
$$h = \tilde{h} + h' \qquad T = \tilde{T} + T' \tag{3.1-3}$$

and so on. We do not need to average the potential, in this case, since it is fixed, e.g. by the potential of the black hole.

3.2 Continuity equation

When we substitute the above into the continuity equation, we obtain

$$\frac{\partial}{\partial t}(\bar{\rho} + \rho') + \frac{\partial}{\partial x_i}(\rho\tilde{v}_i + \rho v_i') = 0 \qquad (3.2\text{-}1)$$

Taking the average of this equation gives

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\tilde{v}_i) = 0 \qquad (3.2\text{-}2)$$

The averaged equation of continuity has the same form as the non-averaged version – mass is conserved in the mean flow.

3.3 Momentum equation

The momentum equation becomes:

$$\frac{\partial}{\partial t}(\rho \tilde{v}_i + \rho v_i') + \frac{\partial}{\partial x_j}[\rho(\tilde{v}_i + v_i')(\tilde{v}_j + v_j')]$$

$$= -\frac{\partial}{\partial x_i}(\bar{p} + p') + \frac{\partial}{\partial x_j}[\bar{\sigma}_{ij} + \sigma_{ij}'] - \rho\frac{\partial \phi_G}{\partial x_i} \qquad (3.3\text{-}1)$$

In constructing an average of this equation, we use

$$\langle \rho v_i' \rangle = 0 \qquad \langle p' \rangle = 0 \qquad \langle \sigma_{ij}' \rangle = 0$$

$$\begin{aligned} \langle \rho(\tilde{v}_i + v_i')(\tilde{v}_j + v_j') \rangle &= \langle \rho \tilde{v}_i \tilde{v}_j \rangle + \langle \rho \tilde{v}_i v_j' \rangle \\ &\quad + \rho \langle v_i' \tilde{v}_j \rangle + \langle \rho v_i' v_j' \rangle \\ &= \bar{\rho} \tilde{v}_i \tilde{v}_j + 0 + 0 + \langle \rho v_i' v_j' \rangle \end{aligned} \tag{3.3-2}$$

so that the momentum equation becomes:

$$\begin{aligned} &\frac{\partial}{\partial t}(\bar{\rho}\tilde{v}_i) + \frac{\partial}{\partial x_j}[\bar{\rho}\tilde{v}_i\tilde{v}_j + \langle \rho v_i' v_j' \rangle] \\ &= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\sigma}_{ij}}{\partial x_j} - \bar{\rho}\frac{\partial \phi_G}{\partial x_i} \end{aligned} \tag{3.3-3}$$

This introduces the non-zero Reynolds stresses

$$t_{ij}^{R} = -\langle \rho v_i' v_j' \rangle \tag{3.3-4}$$

which is the negative of the j^{th} component of the flux of turbulent momentum $(\rho v_i')$.

Equivalently, $-t_{ij}^{R}$ is the mass-weighted correlation of the turbulent velocity components. This term is fundamental to all turbulent flows and describes the turbulent diffusion of momentum in such flows.

The momentum equation may be put in the form:

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{v}_i) + \frac{\partial}{\partial x_j}[\bar{\rho}\tilde{v}_i\tilde{v}_j] = -\frac{\partial\bar{p}}{\partial x_i} + \frac{\partial t^R_{ij}}{\partial x_j} + \frac{\partial\bar{\sigma}_{ij}}{\partial x_j} - \bar{\rho}\frac{\partial\phi_G}{\partial x_i} \qquad (3.3\text{-}5)$$

and the Reynolds stresses appear as an additional force driving the mean motion.

In the above equations, the gravitational potential ϕ_G is regarded as prescribed (e.g. by the potential of the central black hole) so that no averaging of it is required.

3.4 Importance of Reynolds stress in high Reynolds number flows

This formalism introduces two additional terms that we do not usually deal with in classical gas dynamics applications

$$\langle \rho v_i' v_j' \rangle \qquad \text{and} \qquad \sigma_{ij} \tag{3.4-1}$$

In high Reynolds number flows, we can ignore the contribution of $\bar{\sigma}_{ij}$ to the momentum, and we represent the mean momentum equations as

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{v}_i) + \frac{\partial}{\partial x_j}[\bar{\rho}\tilde{v}_i\tilde{v}_j] = -\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \langle \rho v_i' v_j' \rangle}{\partial x_j} - \bar{\rho}\frac{\partial \phi_G}{\partial x_i} \tag{3.4-2}$$

For this reason the Reynolds stress is associated with the concept of *turbulent viscosity*, although that it is not a term which I personally like and which can also lead to misunderstandings of the physics.

3.5 The problem of closure

The fundamental problem in modelling turbulent flows is that it is not obvious what has to be done to express the Reynolds stresses in terms of the fundamental dynamical variables. This is known as the ***Problem of Closure*** and has concerned hydrodynamicists since the 1900s. It is one of the outstanding theoretical physics problems. We shall see one approach to this when we discuss accretion discs.

One approach to closure that has been used is to specify a so-called ***turbulent viscosity*** and to put

$$t_{ij}^{R} = \bar{\rho}\nu_t \tilde{s}_{ij} \qquad (3.5\text{-}1)$$

In this case the turbulent viscosity is written as

$$\nu_t \approx l_t v_t \qquad (3.5\text{-}2)$$

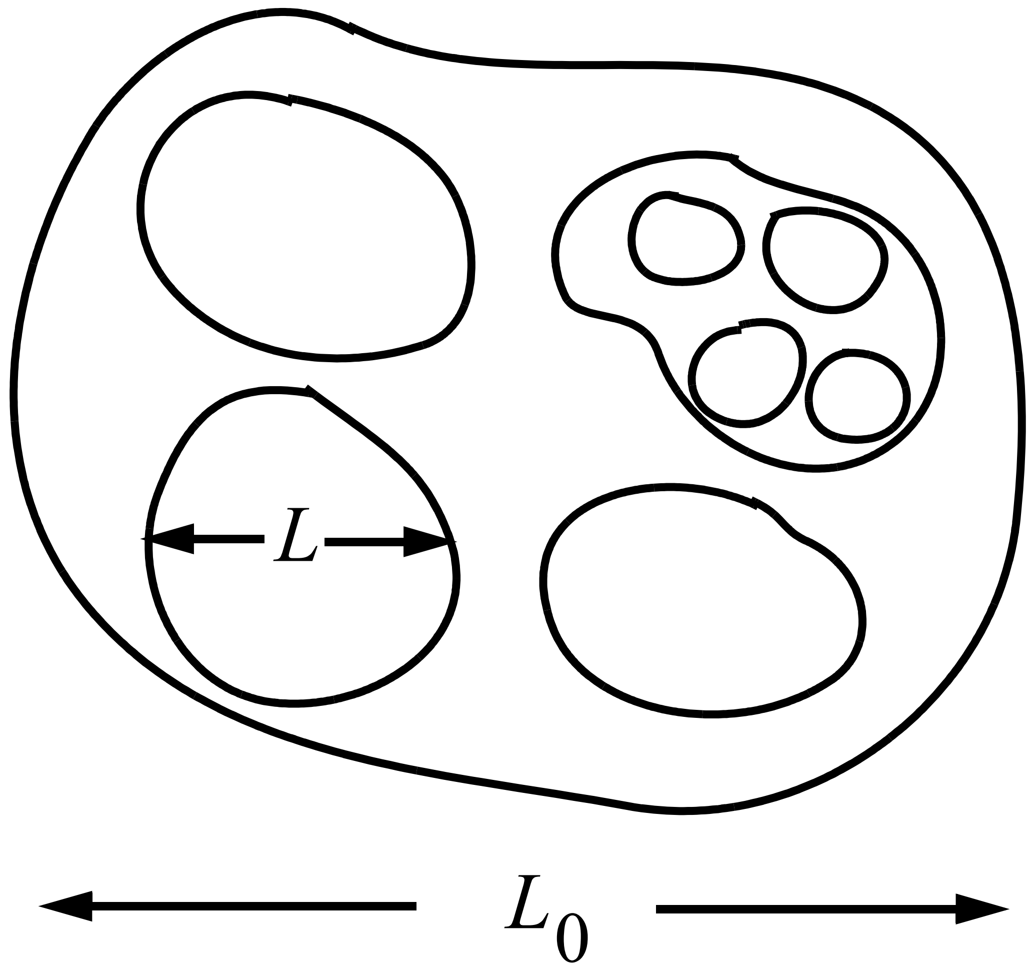
where l_t and v_t are appropriate turbulent length and velocity scales. That is the momentum transport associated with the turbulence is treated as having some of the features of momentum transport associated with molecular processes.

Note the distinction between molecular and turbulent viscosity. The former is usually small; the latter can be significant.

4 Characterisation of turbulent flows

4.1 The turbulent cascade

One of the features of fluid dynamic turbulence is the existence of the turbulent cascade. We regard turbulence as being constructed of *turbulent eddies* and that the energy in these eddies is dissipated on the *eddy turnover timescale*. What happens to the energy – it makes smaller eddies, of course!



Eddies in turbulent flow

Let v_t^0 be the turbulent velocity in the largest eddy of size L_0, then the kinetic energy density is $\frac{1}{2}\rho(v_t^0)^2$ and the eddy ***turnover time*** is $\frac{L_0}{v_t^0}$. The volume rate of energy dissipation associated with this turnover time is:

$$\Lambda_t \sim \frac{1}{2}\frac{\rho(v_t^0)^2}{(L_0/v_t^0)} \sim \frac{1}{2}\frac{\rho(v_t^0)^3}{L_0} \qquad (4.1\text{-}1)$$

Units are energy per unit time per unit volume.

In incompressible flow, this energy ends up in smaller eddies since there are no wave modes by which it can be radiated away. With a turbulent velocity V_t and length scale L, the volume rate of dissipation is:

$$\frac{1}{2}\frac{\rho v_t^3}{L} \tag{4.1-2}$$

This is equal to the volume rate of energy dissipation resulting from the largest eddies, so that

$$\frac{1}{2}\rho\frac{v_t^3}{L} \sim \frac{1}{2}\rho\frac{(v_t^0)^3}{L_0}$$

$$\Rightarrow v_t \sim v_t^0\left(\frac{L}{L_0}\right)^{1/3} \qquad (4.1\text{-}3)$$

This is a classic result for incompressible hydrodynamic turbulence. The numerical situation is different for compressible turbulence and for turbulence in a magnetised fluid but the general physical principles are the same.

4.2 The eventual fate of the cascaded energy

We have just described what is referred to as a turbulent cascade. The energy which ends up on smaller and smaller scales is eventually dissipated by viscosity as we shall now show. First note, however, that the turbulent kinetic energy density is dominated by the large scale, since the kinetic energy density associated with each eddy,

$$T \sim \frac{1}{2}\rho v_t^2 \sim \frac{1}{2}\rho (v_t^0)^2 \left(\frac{L}{L_0}\right)^{2/3} . \qquad (4.2\text{-}1)$$

Consider the contribution of each eddy to the dissipation $\sigma_{ij} v_{i,j}$. We have

$$\sigma_{ij} \sim 2\eta S_{ij} + \zeta v_{k,k} \delta_{ij} \sim \eta \frac{v_t}{L} \tag{4.2-2}$$

since the coefficients of viscosity have the same order of magnitude. Hence, the dissipation of turbulent energy into heat in an eddy of size L is given by:

$$\sigma_{ij} v_{i,j} \sim \eta \left(\frac{v_t}{L}\right)^2 \sim \eta \left(\frac{v_t^0}{L_0}\right)^2 \left(\frac{L}{L_0}\right)^{-4/3} \tag{4.2-3}$$

The volume rate of dissipation goes up as the length scale of the eddy decreases. Hence the rate of dissipation balances the rate of increase of energy within the eddy, when

$$\eta\left(\frac{v_t^0}{L_0}\right)^2\left(\frac{L}{L_0}\right)^{-4/3} \sim \frac{1}{2}\frac{\rho(v_t^0)^3}{L_0} \qquad (4.2\text{-}4)$$

i.e. when the length scale is small enough that

$$\left(\frac{L}{L_0}\right)^{4/3} \sim \left(\frac{\eta}{\rho}\right)(L_0 v_t^0)^{-1} = \frac{\nu}{L_0 v_t^0} = \frac{1}{R_t} \qquad (4.2\text{-}5)$$

where R_t is the Reynolds number of the turbulence.

Now, the coefficient of kinematic viscosity is given to order of magnitude by

$$\nu \sim L_{\mathrm{mfp}} c_s . \tag{4.2-6}$$

Thus, the dissipative length scale L_d, is determined from

$$\left(\frac{L}{L_0}\right)^{4/3} \sim \left(\frac{L_{\mathrm{mfp}}}{L_0}\right)\left(\frac{v_t^0}{c_s}\right)^{-1} \tag{4.2-7}$$

Normally, the mean free path is much smaller than the typical length scale and the turbulent velocity may range (typically) from about 0.1 times the sound speed, to the sound speed. Thus in general L_d is a very small length compared to the largest relevant length scale.

The above describes the characteristics of all turbulence:

- An energy containing scale which determines the eventual dissipation rate
- An inertial range in which the velocity scales with the size of the eddy. This is the turbulent cascade
- A dissipative scale on which the turbulence is dissipated.

4.3 A famous poem

The above physics is neatly encapsulated in the following famous poem:

“Larger whirls have lesser whirls that feed on their velocity

And lesser whirls have smaller whirls

And so on to viscosity”

This has been corrupted to the following:

"Larger fleas have lesser fleas upon their backs to bite'em
And lesser fleas have smaller fleas
And so on ad infinitum"

5 The generation and dissipation of turbulent kinetic energy

5.1 Development of the TKE equation

It is important in any turbulent flow to know how turbulent energy is generated. The way we determine this is to construct an equation for the turbulent kinetic energy.

The turbulent kinetic energy is defined by:

$$\mathrm{TKE} = \left\langle \frac{1}{2}\rho v'^2 \right\rangle \qquad (5.1\text{-}1)$$

We derive an equation for the TKE by first forming an equation for the kinetic energy. This is formed by taking the scalar product of the momentum equations with the velocity. That is,

$$v_i\left[\rho\frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j}\right] = -v_i\frac{\partial p}{\partial x_i} + v_i\frac{\partial \sigma_{ij}}{\partial x_j} - v_i\frac{\partial \phi_G}{\partial x_i} \qquad (5.1\text{-}2)$$

Using,

$$v_i \frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) \qquad v_i \frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{2}v^2\right) \tag{5.1-3}$$

and the equation of continuity, we can put the above equation in the form:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = -v_i \frac{\partial p}{\partial x_i} + v_i \frac{\partial \sigma_{ij}}{\partial x_j} - v_i \frac{\partial \phi_G}{\partial x_i} \tag{5.1-4}$$

We now take the average of this equation using:

$$
\begin{aligned}
\left\langle \frac{1}{2}\rho v^2 \right\rangle &= \frac{1}{2}\bar{\rho}\tilde{v}^2 + \left\langle \frac{1}{2}\rho v'^2 \right\rangle \\
&= \text{Mean Kinetic Energy} + \text{TKE} \\
\left\langle \frac{1}{2}\rho v^2 v_j \right\rangle &= \frac{1}{2}\bar{\rho}\tilde{v}^2\tilde{v}_j + \langle \rho v_i' v_j' \rangle \tilde{v}_i + \left\langle \frac{1}{2}\rho v'^2 \right\rangle \tilde{v}_j \\
&\quad + \left\langle \frac{1}{2}\rho v'^2 v_j' \right\rangle
\end{aligned}
\tag{5.1-5}
$$

and

$$-\left\langle v_i \frac{\partial p}{\partial x_i} \right\rangle = -\tilde{v}_i \frac{\partial \bar{p}}{\partial x_i} - \left\langle v_i' \frac{\partial p}{\partial x_i} \right\rangle$$

$$\left\langle v_i \frac{\partial \sigma_{ij}}{\partial x_j} \right\rangle = \tilde{v}_i \frac{\partial \bar{\sigma}_{ij}}{\partial x_i} + \left\langle v_i' \frac{\partial \sigma_{ij}}{\partial x_j} \right\rangle \qquad (5.1\text{-}6)$$

$$-\left\langle \rho v_i \frac{\partial \phi_G}{\partial x_i} \right\rangle = -\bar{\rho} \tilde{v}_i \frac{\partial \phi_G}{\partial x_i}$$

Thus the averaged equation for the total kinetic energy is:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\bar{\rho}\tilde{v}^2 + \langle\frac{1}{2}\rho v'^2\rangle\right]$$

$$+ \frac{\partial}{\partial x_j}\left[\frac{1}{2}\bar{\rho}\tilde{v}^2\tilde{v}_j + \langle\rho v_i' v_j'\rangle\tilde{v}_i\right]$$

$$+ \frac{\partial}{\partial x_j}\left[\langle\frac{1}{2}\rho v'^2\rangle\tilde{v}_j + \langle\frac{1}{2}\rho v'^2 v_j'\rangle\right] \qquad (5.1\text{-}7)$$

$$= -\tilde{v}_i\frac{\partial\bar{p}}{\partial x_i} - \langle v_i'\frac{\partial p}{\partial x_i}\rangle + \tilde{v}_i\frac{\partial\bar{\sigma}_{ij}}{\partial x_i} + \langle v_i'\frac{\partial\sigma_{ij}}{\partial x_j}\rangle - \bar{\rho}\tilde{v}_i\frac{\partial\phi_G}{\partial x_i}$$

The next step is to take the scalar product of the velocity with the averaged momentum equation. This enables us to subtract out the parts of the above equation relating to the mean kinetic energy. We take the averaged momentum equation in the form:

$$\bar{\rho}\frac{\partial\tilde{v}_i}{\partial t} + \bar{\rho}\tilde{v}_j\frac{\partial\tilde{v}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\langle\rho v_i' v_j'\rangle = -\frac{\partial\bar{p}}{\partial x_i} + \frac{\partial\bar{\sigma}_{ij}}{\partial x_j} - \bar{\rho}\frac{\partial\phi_G}{\partial x_i} \qquad (5.1\text{-}8)$$

The scalar product of $\tilde{v}_i$ with this equation is:

$$\bar{\rho}\tilde{v}_i\frac{\partial\tilde{v}_i}{\partial t} + \bar{\rho}\tilde{v}_j\tilde{v}_i\frac{\partial\tilde{v}_i}{\partial x_j} + \tilde{v}_i\frac{\partial}{\partial x_j}\langle\rho v_i' v_j'\rangle$$

$$= -\tilde{v}_i\frac{\partial\bar{p}}{\partial x_i} + \tilde{v}_i\frac{\partial\bar{\sigma}_{ij}}{\partial x_j} - \bar{\rho}\tilde{v}_i\frac{\partial\phi_G}{\partial x_i} \qquad (5.1\text{-}9)$$

We rearrange the terms on the left of this equation in a similar way to the unaveraged equation, using this time, the mean continuity equation.

The result is:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{\rho}\tilde{v}^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\bar{\rho}\tilde{v}^2\tilde{v}_j\right) + \tilde{v}_i\frac{\partial}{\partial x_j}\langle\rho v_i' v_j'\rangle$$

$$= -\tilde{v}_i\frac{\partial\bar{p}}{\partial x_i} + \tilde{v}_i\frac{\partial\bar{\sigma}_{ij}}{\partial x_j} - \bar{\rho}\tilde{v}_i\frac{\partial\phi_G}{\partial x_i} \tag{5.1-10}$$

The next step is to subtract this equation from the averaged energy equation. Let us put the equations together so that the terms that cancel can be seen clearly:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{1}{2}\bar{\rho}\tilde{v}^2 + \left\langle\frac{1}{2}\rho v'^2\right\rangle\right] \\
&+ \frac{\partial}{\partial x_j}\left[\frac{1}{2}\bar{\rho}\tilde{v}^2\tilde{v}_j + \langle\rho v_i' v_j'\rangle\tilde{v}_i\right] \\
&+ \frac{\partial}{\partial x_j}\left[\left\langle\frac{1}{2}\rho v'^2\right\rangle\tilde{v}_j + \left\langle\frac{1}{2}\rho v'^2 v_j'\right\rangle\right] \\
&= -\tilde{v}_i\frac{\partial\bar{p}}{\partial x_i} - \left\langle v_i'\frac{\partial p}{\partial x_i}\right\rangle + \tilde{v}_i\frac{\partial\bar{\sigma}_{ij}}{\partial x_i} + \left\langle v_i'\frac{\partial\sigma_{ij}}{\partial x_j}\right\rangle - \bar{\rho}\tilde{v}_i\frac{\partial\phi_G}{\partial x_i}
\end{aligned}
\qquad (5.1\text{-}11)
$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{\rho}\tilde{v}^2\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\bar{\rho}\tilde{v}^2\tilde{v}_i\right) + \tilde{v}_i\frac{\partial}{\partial x_j}\langle\rho v_i' v_j'\rangle$$

$$= -\tilde{v}_i\frac{\partial\bar{p}}{\partial x_i} + \tilde{v}_i\frac{\partial\bar{\sigma}_{ij}}{\partial x_j} - \bar{\rho}\tilde{v}_i\frac{\partial\phi_G}{\partial x_i} \qquad (5.1\text{-}12)$$

The resulting TKE equation is:

$$\frac{\partial}{\partial t}\left[\langle\frac{1}{2}\rho v'^2\rangle\right]$$

$$+\frac{\partial}{\partial x_j}\left[\langle\rho v_i' v_j'\rangle\tilde{v}_i+\langle\frac{1}{2}\rho v'^2\rangle\tilde{v}_j+\langle\frac{1}{2}\rho v'^2 v_j'\rangle\right] \qquad (5.1\text{-}13)$$

$$-\tilde{v}_i\frac{\partial}{\partial x_j}\langle\rho v_i' v_j'\rangle = -\langle v_i'\frac{\partial p}{\partial x_i}\rangle+\langle v_i'\frac{\partial\sigma_{ij}}{\partial x_j}\rangle$$

The following terms give:

$$\frac{\partial}{\partial x_j}[\langle \rho v_i' v_j' \rangle \tilde{v}_i] - \tilde{v}_i \frac{\partial}{\partial x_j} \langle \rho v_i' v_j' \rangle = \langle \rho v_i' v_j' \rangle v_{i,j} \tag{5.1-14}$$

The TKE equation becomes:

$$\frac{\partial}{\partial t}\left[\langle \frac{1}{2}\rho v'^2 \rangle\right] + \frac{\partial}{\partial x_j}\left[\langle \frac{1}{2}\rho v'^2 \rangle \tilde{v}_j + \langle \frac{1}{2}\rho v'^2 v_i' \rangle\right] = -\langle v_i' \frac{\partial p}{\partial x_i} \rangle + \langle v_i' \frac{\partial \sigma_{ij}}{\partial x_j} \rangle - \langle \rho v_i' v_j' \rangle v_{i,j} \tag{5.1-15}$$

5.2 Interpretation of terms

The meanings of the various turbulent terms are as follows:

$$\begin{aligned}
\langle \tfrac{1}{2}\rho v'^2 \rangle &= \text{Turbulent kinetic energy} \\
\langle \rho v_i' v_j' \rangle \tilde{v}_i &= \text{Rate of work done by turbulent stresses on mean velocity} \\
\langle \tfrac{1}{2}\rho v'^2 \rangle \tilde{v}_i &= \text{Flux of TKE due to mean velocity} \\
\langle \tfrac{1}{2}\rho v'^2 v_i' \rangle &= \text{Turbulent flux of TKE}
\end{aligned} \tag{5.2-1}$$

and

$$-\langle v_i' \frac{\partial p}{\partial x_i} \rangle + \langle v_i' \frac{\partial \sigma_{ij}}{\partial x_j} \rangle = \text{Work done by pressure and viscous stresses on turbulent fluctuations} \qquad (5.2\text{-}2)$$

$$-\langle \rho v_i' v_j' \rangle v_{i,j} = \text{Rate of production of turbulent energy}$$

Note also that:

$$\langle \rho v_i' v_j' \rangle v_{i,j} \qquad (5.2\text{-}3)$$

only involves the symmetric components of $v_{i,j}$ since $\langle \rho v_i' v_j' \rangle$ is symmetric.

We put for the symmetric part of $\tilde{v}_{i,j}$:

$$\begin{aligned}\tilde{v}_{(i,j)} &= \frac{1}{2}\left(\tilde{v}_{i,j} + \tilde{v}_{j,i} - \frac{2}{3}\tilde{v}_{k,k}\delta_{ij}\right) + \frac{1}{3}\tilde{v}_{k,k}\delta_{ij} \\ &= \tilde{s}_{ij} + \frac{1}{3}\tilde{v}_{k,k}\delta_{ij}\end{aligned} \qquad (5.2\text{-}4)$$

where $\tilde{s}_{ij}$ is the shear associated with the mean velocity.

The TKE equation then becomes:

$$\frac{\partial}{\partial t}\left[\langle\frac{1}{2}\rho v'^2\rangle\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{2}\langle\rho v'^2\rangle\tilde{v}_j + \langle\frac{1}{2}\rho v'^2 v_j'\rangle\right]$$

$$+ \frac{2}{3}\left[\langle\frac{1}{2}\rho v'^2\rangle\right]\tilde{v}_{k,k} \qquad (5.2\text{-}5)$$

$$= -\langle v_i'\frac{\partial p}{\partial x_i}\rangle + \langle v_i'\frac{\partial\sigma_{ij}}{\partial x_j}\rangle - \langle\rho v_i' v_j'\rangle\tilde{s}_{ij}$$

One final rearrangement

Put

$$\left\langle v_i' \frac{\partial \sigma_{ij}}{\partial x_j} \right\rangle = \frac{\partial}{\partial x_j} \langle \sigma_{ij} v_i \rangle - \langle v'_{i,j} \sigma_{ij} \rangle$$

$$-\left\langle v_i' \frac{\partial p}{\partial x_i} \right\rangle = -\frac{\partial}{\partial x_i} \langle p v_i' \rangle + \langle p v'_{i,i} \rangle \qquad (5.2\text{-}6)$$

Our final form of the TKE equation is then:

$$\frac{\partial}{\partial t}\left[\langle\frac{1}{2}\rho v'^2\rangle\right]$$

$$+\frac{\partial}{\partial x_j}\left[\frac{1}{2}\langle\rho v'^2\rangle\tilde{v}_i+\langle\frac{1}{2}\rho v'^2 v_i'\rangle+\langle p v_j'\rangle-\langle\sigma_{ij}v_i\rangle\right] \tag{5.2-7}$$

$$+\frac{2}{3}\left[\langle\frac{1}{2}\rho v'^2\rangle\right]\tilde{v}_{k,k}$$

$$=\langle p v'_{i,i}\rangle-\langle v'_{i,j}\sigma_{ij}\rangle-\langle\rho v_i' v_j'\rangle\tilde{s}_{ij}$$

6 The dissipation of TKE into internal energy

6.1 The averaged internal energy equation

The other energy equation we need to bring into play here is the equation for the internal energy density, As we shall see this complements the TKE equation that we have just derived.

In unaveraged from this is:

$$\rho k T \frac{ds}{dt} = \sigma_{ij} v_{i,j} - q_{i,i} \qquad (6.1\text{-}1)$$

Using the relations:

$$\begin{aligned} \rho k T ds &= d\varepsilon - h d\rho \\ \rho k T ds &= \rho dh - dp \end{aligned} \qquad (6.1\text{-}2)$$

implies:

$$\rho kT\frac{\partial s}{\partial t} = \frac{\partial \varepsilon}{\partial t} - \frac{(\varepsilon + P)}{\rho}\frac{\partial \rho}{\partial t} = \frac{\partial \varepsilon}{\partial t} - h\frac{\partial \rho}{\partial t}$$

$$\rho kTv_i\frac{\partial s}{\partial x_i} = \rho v_i\frac{\partial h}{\partial x_i} - v_i\frac{\partial p}{\partial x_i} \qquad (6.1\text{-}3)$$

$$\Rightarrow \rho kT\frac{ds}{dt} = \frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}(\rho h v_i) - v_i\frac{\partial p}{\partial x_i}$$

This energy equation then becomes:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}(\rho h v_i) - v_i\frac{\partial p}{\partial x_i} = \sigma_{ij}v_{i,j} - q_{i,i} \qquad (6.1\text{-}4)$$

The averaged form of this equation is:

$$\frac{\partial\bar{\varepsilon}}{\partial t}+\frac{\partial}{\partial x_i}[\bar{\rho}\tilde{h}\tilde{v}_i+\langle\rho h' v'_i\rangle]-\langle v_i\frac{\partial p}{\partial x_i}\rangle = \langle\sigma_{ij}v_{i,j}\rangle-\bar{q}_{i,i} \qquad (6.1\text{-}5)$$

Again, we write

$$\langle v_i\frac{\partial p}{\partial x_i}\rangle = \tilde{v}_i\frac{\partial\bar{p}}{\partial x_i}+\langle v_i'\frac{\partial p}{\partial x_i}\rangle \qquad (6.1\text{-}6)$$

$$\langle\sigma_{ij}v_{i,j}\rangle = \bar{\sigma}_{ij}\tilde{v}_{i,j}+\langle\sigma_{ij}v'_{i,j}\rangle$$

The averaged energy equation is therefore:

$$\frac{\partial\bar{\varepsilon}}{\partial t} + \frac{\partial}{\partial x_i}[\bar{\rho}\tilde{h}\tilde{v}_i + \langle \rho h' v'_i \rangle] - \tilde{v}_i \frac{\partial \bar{p}}{\partial x_i}$$

$$= \langle v_i' \frac{\partial p}{\partial x_i} \rangle + \bar{\sigma}_{ij}\tilde{v}_{i,j} - \bar{q}_{i,i} + \langle \sigma_{ij} v'_{i,j} \rangle \qquad (6.1\text{-}7)$$

6.2 Interpretation of terms

The interpretation of the terms in this equation are:

$$\frac{\partial\bar{\varepsilon}}{\partial t} = \text{Time rate of change of mean internal energy}$$

$$\bar{\rho}\tilde{h}\tilde{v}_i = \text{Mean heat flux}$$

$$\langle \rho h' v'_i \rangle = \text{Heat flux due to turbulent diffusion} \qquad (6.2\text{-}1)$$

$$\tilde{v}_i \frac{\partial\bar{p}}{\partial x_i} = \text{Mean rate of work of pressure gradient on fluid}$$

The terms on the right hand side:

$$\langle v_i' \frac{\partial p}{\partial x_i} \rangle = \text{Rate of work on turbulent velocity by pressure}$$

$$\bar{\sigma}_{ij} \tilde{v}_{i,j} = \text{Rate of viscous dissipation due to mean velocity} \qquad (6.2\text{-}2)$$

$$\bar{q}_{i,i} = \text{Mean heat flux}$$

$$\langle \sigma_{ij} v'_{i,j} \rangle = \text{Viscous dissipation due to turbulent velocity}$$

In a turbulent fluid of high Reynolds number the important dissipation term is the last one: $\langle \sigma_{ij} v'_{i,j} \rangle$ – the rate of dissipation resulting from viscous stresses acting on the turbulent velocity.

This equation tells us that the internal energy density of a fluid can increase as a result of the dissipation due to viscosity, that heat is removed from a given volume via mean heat flux due to the bulk motion of the fluid $(\bar{\rho}\tilde{h}\tilde{v}_i)$, turbulent diffusion $(\langle \rho h' v'_i \rangle)$ and molecular heat flux $(\bar{q}_{i,i})$. In a turbulent fluid at rest in the mean, the turbulent diffusive heat flux is the most important way of diffusing heat.

6.3 Relation to the TKE equation

In the TKE equation the term

$$-\langle v'_{i,j} \sigma_{ij} \rangle \qquad (6.3\text{-}1)$$

appears with the opposite sign to the corresponding term in the energy equation. In the TKE equation this term represents a sink; in the energy equation it represents a source of internal energy. The physics of this is that TKE is being generated as a result of the term

$$-\langle \rho v_i' v_j' \rangle \tilde{s}_{ij} \qquad (6.3\text{-}2)$$

and is being dissipated as a result of the term

$$-\langle v'_{i,j} \sigma_{ij} \rangle \qquad (6.3\text{-}3)$$

The last term (with the opposite sign) becomes a source term in the energy equation.

6.4 Turbulence in a quasi steady state

The presence of the term $-\langle \sigma_{ij} v_{i,j}' \rangle$ on the right hand side of the above equation shows clearly that the turbulent kinetic energy is viscously dissipated. However, other things happen as well. The presence of the term $\langle \frac{1}{2} \rho v'^2 v'_i \rangle$ also shows that it can be diffused away from the point where it is created. One of the single most important terms in the above is the term $-\langle \rho v'_i v'_j \rangle \tilde{s}_{ij}$ which is the rate at which the Reynolds stresses work on the mean shear and is the major source term for turbulence.

6.5 Difference from standard accretion disc theory

Note that in just about every treatment of accretion discs it is assumed that $-\langle\rho v'_i v'_j\rangle\tilde{s}_{ij}$ is the rate at which turbulence is dissipated into heat. This is only true if we have the balance:

$$\langle\sigma_{ij}v_{i,j}\rangle = -\tilde{s}_{ij}\langle\rho v'_i v'_j\rangle \qquad (6.5\text{-}1)$$

This is generally not locally true, but it is reasonable to assume that it is true in an averaged sense. The reason for this confusion in the literature is that the term *turbulent viscosity* which is really only in a model for the turbulent Reynolds stress but which has been used to treat the turbulent viscosity in a similar way to real viscosity. As we have seen the Reynolds stress governs the ***production*** of turbulent energy