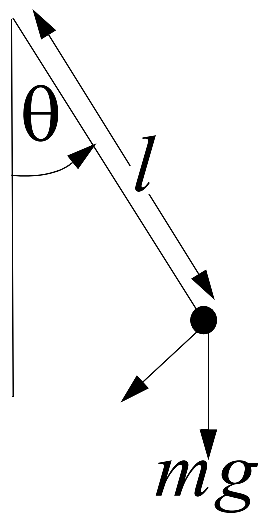
# The Sedov Solution

## 1 Similarity solutions




Equation of motion of pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \qquad (1)$$

Suppose solution of the form:

$$\theta = f(\theta_0, t, l, g) \tag{2}$$

Since $f$ is a dimensionless function it cannot depend upon the dimensions of $t$, $l$, or $g$. Hence we need to combine $t$, $l$ and $g$ into some dimensionless combination, the only possible one being $t\sqrt{\frac{g}{l}}$ and the solution is of the form

$$\theta = f\left(\theta_0, t\sqrt{\frac{g}{l}}\right) \tag{3}$$

In this problem the variable $t\sqrt{\frac{g}{l}}$ is a similarity variable. We construct analogous variables in constructing useful solutions to the Euler equations.

# 2 Spherically symmetric hydrodynamics

## 2.1General equations

Take the spherically symmetric Euler equations:

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \qquad (4)$$

$$\frac{\partial}{\partial t}(p\rho^{-\gamma}) + v\frac{\partial}{\partial r}(p\rho^{-\gamma}) = 0$$

The last equation is equivalent to

$$p = K(s)\rho^{\gamma} \qquad \frac{dK(s)}{dt} = \frac{\partial}{\partial t}K(s) + v\frac{\partial}{\partial r}K(s) = 0 \qquad (5)$$

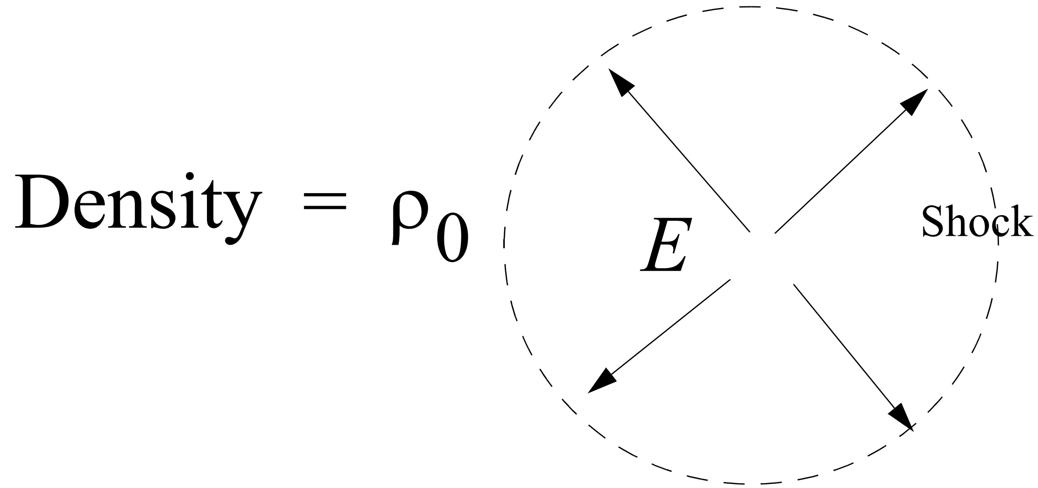
The independent variables in this problem are $r$ and $t$. In general we need to solve partial differential equations in $r$ and $t$ to obtain a solution.

In the self-similar approach we look for solutions of one variable $rt^{\lambda}$, leading to ordinary differential equations. However, we need some physical basis for choosing $\lambda$.

## 2.2 Point explosion in a uniform medium

Terrestrial application: Analysis of Atom Bombs

Astrophysical application: 2nd phase of supernova remnants




We expect the general features of the explosion to be a spherical expansion preceded by a shock wave advancing into the undisturbed gas.

## 2.3Dimensional analysis

In order to find an appropriate similarity variable, we look at the dimensions of the parameters involved.

$$[E] = [\rho][V^2][L^3]$$

$$\left[\frac{E}{\rho_0}\right] = [L^5 T^{-2}] = [L T^{-2/5}]^5 \quad (6)$$

$$\left[\left(\frac{E}{\rho_0}\right)^{1/5}\right] = L T^{-2/5}$$

Thus an appropriate (dimensionless) similarity variable for this problem is

$$\xi \propto \left(\frac{E}{\rho_0}\right)^{-1/5} r t^{-2/5} \quad (7)$$

We aim to construct the similarity variable so that the radius of the shock produced by the explosion is at $\xi = 1$. We therefore take

$$\xi = \beta\left(\frac{E}{\rho_0}\right)^{-1/5} r t^{-2/5} \tag{8}$$

where $\beta$ is to be determined later.

It is not envisaged that all features of such an explosion are defined by taking the fluid variables to be functions of $\xi$. However, what is envisaged is that after some time, the resultant flow will settle down to a similarity solution in which the parameters $E$, $\rho$ and the variables $r$ and $t$ are combined into one similarity variable. This is a general feature of similarity solutions.

# 3 Self-similar form of the Euler equations

## 3.1Self similar forms of variables and derivatives

Guided by the dimensions of the dynamica variables, we take:

$$\begin{aligned} \rho &= \rho_0 G(\xi) \\ v &= A\frac{r}{t}U(\xi) \\ c_s^2 &= B\frac{r^2}{t^2}Z(\xi) \end{aligned} \qquad (9)$$

The constants $A$ and $B$ will be chosen to make the ensuing equations as simple as possible. This form of the fluid variables is chosen in such a way that they are dimensionally correct. (Obviously, there is a strong connection between dimensional analysis and self-similarity.)

The momentum equation is expresed in the following form:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial r} = -c_s^2\frac{1}{\rho}\frac{\partial \rho}{\partial r} = -c_s^2\frac{\partial}{\partial r}\ln\rho \qquad (10)$$

The final form of the Euler equations is:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \rho\frac{\partial v}{\partial r} + \frac{2\rho v}{r} &= 0 \\
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + c_s^2\frac{\partial}{\partial r}\ln\rho &= 0 \\
\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\ln(p\rho^{-\gamma}) &= 0
\end{aligned} \tag{11}$$

Since

$$p\rho^{-\gamma} = \frac{1}{\gamma}\left(\frac{\gamma p}{\rho}\right)\rho^{-(\gamma-1)} = \frac{1}{\gamma}c_s^2\rho^{-(\gamma-1)} \tag{12}$$

then the last of the above equations can be written

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)(\ln c_s^2 - (\gamma - 1)\ln\rho) = 0 \tag{13}$$

We adopt the general self-similar variable:

$$\xi = Crt^{\lambda} \tag{14}$$

where $\lambda = -\frac{2}{5}$ in the constant background density case discussed above. Other values of $\lambda$ are relevant when the background density is non-uniform.

W e have to do the following preliminaries, since differentiation of functions of $\xi$ involves differentiation of $\xi$ with respect to $t$ and $r$.

Derivatives of $\xi$

$$\frac{\partial \xi}{\partial r} = C t^{\lambda} = \frac{\xi}{r}$$
$$\frac{\partial \xi}{\partial t} = \lambda C r t^{\lambda - 1} = \lambda \frac{\xi}{t} \tag{15}$$

## *Derivatives of density*

$$\frac{\partial\rho}{\partial r} = \rho_0 G'(\xi)\frac{\xi}{r} \qquad \frac{\partial \ln\rho}{\partial r} = \frac{1}{r}\frac{\xi G'(\xi)}{G(\xi)}$$

$$\frac{\partial\rho}{\partial t} = \rho_0 \frac{\lambda\xi}{t} G'(\xi) \qquad \frac{\partial \ln\rho}{\partial t} = \frac{\lambda}{t}\frac{\xi G'(\xi)}{G(\xi)} \tag{16}$$

## *Derivatives of velocity*

$$\frac{\partial v}{\partial t} = A\frac{r}{t^2}[-U(\xi) + \lambda\xi U'(\xi)]$$

$$\frac{\partial v}{\partial r} = \frac{A}{t}[U(\xi) + \xi U'(\xi)] \tag{17}$$

# Derivatives of sound speed:

Since

$$\ln c_s^2 = \ln(\text{constant}) + 2\ln\frac{r}{t} + \ln Z \qquad (18)$$

then

$$\begin{aligned} \frac{\partial}{\partial t}\ln c_s^2 &= \frac{1}{t}\left[-2 + \lambda\frac{\xi Z'(\xi)}{Z(\xi)}\right] \\ \frac{\partial}{\partial r}\ln c_s^2 &= \frac{1}{r}\left[2 + \frac{\xi Z'(\xi)}{Z(\xi)}\right] \end{aligned} \qquad (19)$$

# 3.2 Euler equations in self-similar form

## 3.2.1 Continuity equation

This is:

$$\frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial r} + \rho\frac{\partial v}{\partial r} + 2\frac{\rho v}{r} = 0 \tag{20}$$

Substituting the expressions for ρ, $v$ and derivatives:

$$\begin{aligned}&\rho_0\frac{\lambda\xi}{t}G'(\xi) + A\frac{r}{t}U(\xi)\rho_0 G'(\xi)\frac{\xi}{r}\\ &+ \rho_0 G(\xi)\frac{A}{t}[U(\xi) + \xi U'(\xi)] + \frac{2}{r}\rho_0 G(\xi)A\frac{r}{t}U(\xi) = 0\end{aligned} \tag{21}$$

The various terms have a common multiplicative factor of $\rho_0 t^{-1}$, which cancels out. The terms can then be combined to give the following equation:

$$[\lambda + AU(\xi)]\xi G'(\xi) + AG(\xi)\xi U'(\xi) + 3AU(\xi)G(\xi) = 0 \quad (22)$$

which is, on dividing by $G(\xi)$

$$[\lambda + AU(\xi)]\frac{\xi G'(\xi)}{G(\xi)} + A\xi U'(\xi) + 3AU(\xi) = 0 \quad (23)$$

Note the appearance of terms such as $\frac{\xi G'(\xi)}{G(\xi)} = \frac{d\ln G(\xi)}{d\ln \xi}$ which often appear when we are dealing with self similar solutions.

### 3.2.2 Momentum equation

We can treat the momentum equation similarly.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + c_s^2\frac{\partial}{\partial r}\ln\rho = 0$$

$$A\frac{r}{t^2}[-U + \lambda\xi U'(\xi)] + A\frac{r}{t}U(\xi)\frac{A}{t}[U + \xi U'(\xi)]$$

$$+ B\frac{r^2}{t^2}Z(\xi)\frac{\xi}{r}\frac{G'(\xi)}{G(\xi)} = 0$$

A factor of $\frac{r}{t^2}$ cancels and we can then collect terms in $U'(\xi)$, $U(\xi)$ and $G'(\xi)$ to obtain the final result:

$$A[\lambda + AU(\xi)]\xi U'(\xi) + AU(\xi)[AU(\xi) - 1] + BZ(\xi)\frac{\xi G'(\xi)}{G(\xi)} = 0 \qquad (24)$$

### 3.2.3 Entropy equation

Similarly, the entropy equation becomes:

$$\frac{\xi Z'(\xi)}{Z(\xi)} - (\gamma - 1)\frac{\xi G'(\xi)}{G(\xi)} + \frac{2(AU(\xi) - 1)}{\lambda + AU(\xi)} = 0 \qquad (25)$$

## 3.3Final form of self-similar equations

The set of self-similar equations is:

$$[\lambda + AU(\xi)]\xi G'(\xi) + AG(\xi)\xi U'(\xi) + 3AU(\xi)G(\xi) = 0$$

$$A[\lambda + AU(\xi)]\xi U'(\xi) + AU(\xi)[AU(\xi) - 1] + BZ(\xi)\frac{\xi G'(\xi)}{G(\xi)} = 0$$

$$\frac{\xi Z'(\xi)}{Z(\xi)} - (\gamma - 1)\frac{\xi G'(\xi)}{G(\xi)} + \frac{2(AU(\xi) - 1)}{\lambda + AU(\xi)} = 0$$

For the point explosion problem in a constant density background, we have $\lambda = -\frac{2}{5}$. Therefore,

$$\left[-\frac{2}{5} + AU(\xi)\right]\xi G'(\xi) + AG(\xi)\xi U'(\xi) + 3AU(\xi)G(\xi) = 0$$

$$A\left[-\frac{2}{5} + AU(\xi)\right]\xi U'(\xi) + \frac{2}{5}U(\xi)\left[\frac{2}{5}U(\xi) - 1\right] + BZ(\xi)\frac{\xi G'(\xi)}{G(\xi)} = 0$$

$$\frac{\xi Z'(\xi)}{Z(\xi)} - (\gamma - 1)\frac{\xi G'(\xi)}{G(\xi)} + \frac{2(AU(\xi) - 1)}{-\frac{2}{5} + AU(\xi)} = 0$$

The above equations are simplified, if we take

$$A = \frac{2}{5} \qquad B = \left(\frac{2}{5}\right)^2 = \frac{4}{25} \tag{26}$$

The final self-similar form is:

$$(U-1)\frac{d\ln G}{d\ln\xi} + \frac{dU}{d\ln\xi} + 3U = 0 \quad \text{(Continuity)}$$

$$(U-1)\frac{dU}{d\ln\xi} + U\left[U - \frac{5}{2}\right] + Z\frac{d\ln G}{d\ln\xi} = 0 \quad \text{(Momentum)} \tag{27}$$

$$\frac{d\ln Z}{d\ln\xi} - (\gamma-1)\frac{d\ln G}{d\ln\xi} + \frac{2(U-5/2)}{U-1} = 0 \quad \text{(Entropy)}$$

# 4 Conditions at outgoing shock

## 4.1 Development of boundary conditions

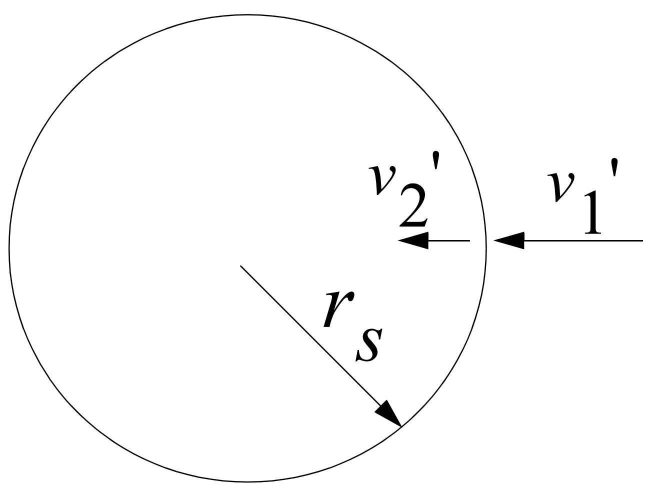
The surface $\xi = 1$ is to represent the outgoing shock wave. Since

$$\xi = \beta\left(\frac{E}{\rho_0}\right)^{-1/5} r t^{-2/5} \qquad (28)$$

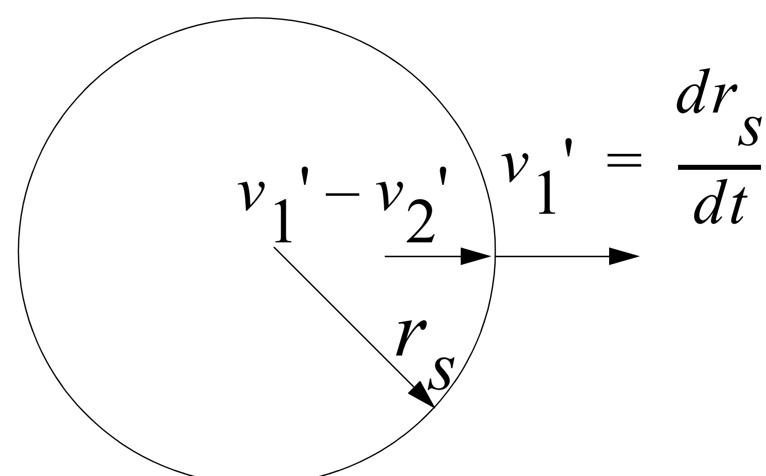
$$\text{Radius of shock} = R_s = \beta^{-1}\left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$$

$$\text{Velocity of shock} = \frac{dR_s}{dt} = \frac{2}{5}\beta^{-1}\left(\frac{E}{\rho_0}\right)^{1/5} t^{-3/5} = \frac{2}{5}\frac{R_s}{t} \qquad (29)$$

Within the constraints of this self-similar solution the only sort of shock that we can contemplate is an infinitely strong shock. If we were to introduce another parameter into the solution corresponding to say, the strength of the shock, then our initial assumption that the solution only depends upon $E$, $\rho_0$, $r$ and $t$ would be invalid and the self-similar solution could no longer be satisfied.




In frame of shock




Frame of stationary gas

We use the relationships for a strong shock:

$$
\begin{aligned}
v_2' &= \frac{\gamma - 1}{\gamma + 1} v_1' \\
\Rightarrow v(r_s) &= v_1' - v_2' \\
&= \frac{2}{\gamma + 1} v_1' = \frac{2}{\gamma + 1}\left(\frac{2}{5}\frac{r}{t}\right)
\end{aligned}
\tag{30}
$$

In view of the self-similar relationship between $v$ and $U$, viz. $v = \frac{2}{5}\frac{r}{t}U(\xi)$, this boundary condition becomes:

$$
U(1) = \frac{2}{\gamma + 1} \tag{31}
$$

Using the density ratio for a strong shock, the density just inside the shock wave is given by:

$$\rho(r_s) = \rho_0 G(1) = \frac{\gamma+1}{\gamma-1}\rho_0$$
$$\Rightarrow G(1) = \frac{\gamma+1}{\gamma-1} \tag{32}$$

In the present notation the relationship between shock velocity, pressure and density is:

$$(v'_1)^2 = \frac{\gamma+1}{2}\frac{p(r_s)}{\rho_0} \Rightarrow p(r_s) = \frac{2}{\gamma+1}\rho_0\left(\frac{dr_s}{dt}\right)^2 \tag{33}$$

Hence, the sound speed just inside the shock is given by:

$$c_s^2(r_s) = \frac{\gamma p(r_s)}{\rho(r_s)} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\left(\frac{dr_s}{dt}\right)^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \times \frac{4}{25}\frac{r_s^2}{t^2} \quad (34)$$

Since

$$c_s^2 = \frac{4}{25}\frac{r^2}{t^2}Z(\xi) \quad (35)$$

then the boundary condition on $Z(\xi)$ is

$$Z(1) = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \quad (36)$$

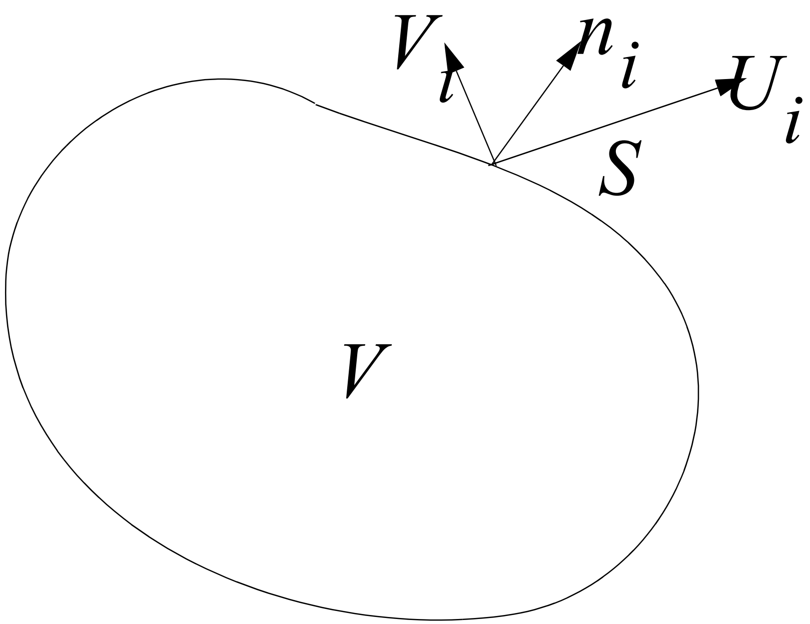
## 4.2Summary of boundary conditions

$$
\begin{aligned}
U(1) &= \frac{2}{\gamma+1} \\
G(1) &= \frac{\gamma+1}{\gamma-1} \\
Z(1) &= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}
\end{aligned}
\qquad (37)
$$

# 5 First integral via energy equation

## 5.1 Conservation laws for a moving surface

When considering conservation laws earlier, we took the surface




enclosing a given volume to be stationary. Let us now consider the case when the enclosing surface is moving. We consider the generic conservation law:

$$\frac{\partial f}{\partial t} + \frac{\partial F_i}{\partial x_i} = 0$$
$$\Rightarrow \frac{\partial}{\partial t}\int_V f d^3x + \int_S F_i n_i dS = 0 \tag{38}$$

corresponding to conservation of a density $f$ with corresponding flux density $F_i$.

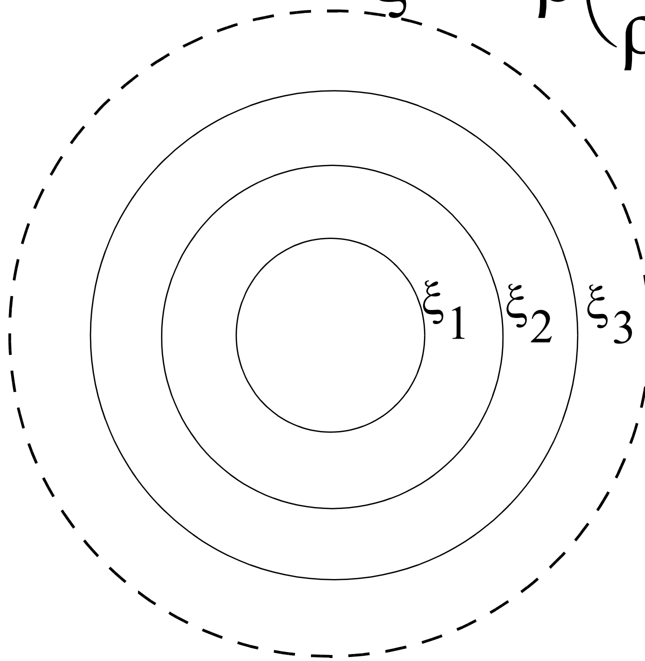
When the surface $S$ is moving with velocity $U_i$ it "sweeps up" $f$ and the corresponding flux *into* $S$ is $fU_i n_i$. Hence the corresponding conservation law is:

$$\frac{\partial}{\partial t}\int_V f d^3x + \int_S F_i n_i dS = \int_S f U_i n_i dS$$
$$\Rightarrow \frac{\partial}{\partial t}\int_V f d^3x + \int_S (F_i - fU_i) n_i dS = 0 \tag{39}$$

## 5.2 Application to the Sedov solution

In our development of the Sedov solution we have the moving surfaces $\xi = \text{constant}$.

$$\xi = \beta\left(\frac{E}{\rho_0}\right)^{-1/5} r t^{-2/5}$$




Energy within $\xi = $ constant:

$$E(\xi) = 4\pi \int_0^{r(\xi)} \rho\left(\frac{1}{2}V^2 + \frac{\varepsilon}{\rho}\right) r^2 dr \tag{40}$$

Now

$$\frac{\varepsilon}{\rho} = \frac{1}{\gamma-1}\frac{P}{\rho} = \frac{1}{\gamma(\gamma-1)}\frac{\gamma P}{\rho} = \frac{1}{\gamma(\gamma-1)}\frac{\gamma P}{\rho} = \frac{1}{\gamma(\gamma-1)}c_s^2 \quad (41)$$

$$\therefore E(\xi) = 4\pi\int_0^{r(\xi)} \rho\left(\frac{1}{2}V^2 + \frac{c_s^2}{\gamma(\gamma-1)}\right) r^2 dr \quad (42)$$

We have

$$
\begin{aligned}
r(\xi) &= \beta^{-1}\left(\frac{E}{\rho_0}\right)^{1/5}\xi t^{2/5} \\
dr &= \beta^{-1}\left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5} d\xi \\
v^2(r, t) &= \frac{4}{25}\frac{r^2}{t^2}U^2(\xi) \\
c_s^2 &= \frac{4}{25}\frac{r^2}{t^2}Z(\xi)
\end{aligned}
\qquad (43)
$$

Therefore the energy becomes:

$$E(\xi) = 4\pi\int_0^{\xi}\rho_0 G(\xi)\left[\frac{1}{2}\frac{4}{25}\frac{r^2}{t^2}U^2(\xi) + \frac{1}{\gamma(\gamma-1)}\frac{4}{25}\frac{r^2}{t^2}Z(\xi)\right] \times \beta^{-3}\left(\frac{E}{\rho_0}\right)^{3/5} t^{6/5}\xi^2 d\xi \quad (44)$$

The terms explicitly involving $r$ and $t$ combine to give:

$$r^2 t^{-4/5} = \beta^{-2}\xi^2\left(\frac{E}{\rho_0}\right)^{2/5} \quad (45)$$

so that the energy is

$$E(\xi) = \frac{16\pi}{25}\beta^{-5}E\int_0^{\xi}\xi^4 G(\xi)\left[\frac{1}{2}U^2(\xi) + \frac{1}{\gamma(\gamma-1)}Z(\xi)\right]d\xi \quad (46)$$

One important aspect of this result is that $E(\xi)$ is independent of the time $t$. We shall use this result shortly. The other important result is that it gives the way of calculating $\beta$. The energy within $\xi = 1$ must be equal to the total energy of the blast, so that $E(1) = E$ and

$$\beta^5 = \frac{16\pi}{25}\int_0^1 \xi^4 G(\xi)\left[\frac{1}{2}U^2(\xi) + \frac{1}{\gamma(\gamma-1)}Z(\xi)\right]d\xi \qquad (47)$$

The parameter $\beta$ can be calculated from this once expressions are obtained for $U(\xi)$ and $Z(\xi)$.

## 5.3 First integral

The conservation of energy applied to the time evolving volume enclosed by $\xi = \text{constant}$ is

$$\frac{\partial}{\partial t}\int_0^{r(\xi)}\left(\varepsilon + \frac{1}{2}\rho V^2\right)d^3x + \int_\Omega\left[\rho V\left(\frac{1}{2}V^2 + h\right) - \left(\varepsilon + \frac{1}{2}\rho V^2\right)\frac{dr(\xi)}{dt}\right]r^2 d\Omega = 0 \qquad (48)$$

Since the energy is independent of time, the surface integral must be zero. The terms in the integrand are independent of angle and therefore the integrand itself must be zero. Thus,

$$\rho v\left(\frac{1}{2}v^2 + h\right) = \left(\varepsilon + \frac{1}{2}\rho v^2\right)\frac{dr(\xi)}{dt}$$

$$\Rightarrow \rho\left[\frac{2}{5}\frac{r}{t}U(\xi)\right]\left[\frac{1}{2}v^2 + h\right] = \frac{2}{5}\frac{r}{t}\left[\varepsilon + \frac{1}{2}\rho v^2\right] \qquad (49)$$

$$\Rightarrow U(\xi)\left[\frac{1}{2}v^2 + \frac{c_s^2}{\gamma - 1}\right] = \frac{\varepsilon}{\rho} + \frac{1}{2}v^2 = \frac{c_s^2}{\gamma(\gamma - 1)} + \frac{1}{2}v^2$$

Using the self-similar substitutions for $v$ and $c_s^2$ gives

$$U(\xi)\left[\frac{1}{2}U^2(\xi) + \frac{1}{\gamma - 1}Z(\xi)\right] = \left[\frac{Z(\xi)}{\gamma(\gamma - 1)} + \frac{1}{2}U^2(\xi)\right] \quad (50)$$

This is easily solved for $Z$ to give:

$$Z(\xi) = \frac{\gamma(\gamma - 1)}{2}\frac{U^2(\xi)[1 - U(\xi)]}{[\gamma U(\xi) - 1]} \quad (51)$$

This is the energy integral!

One can see that this integral automatically satisfies the boundary conditions

$$\begin{aligned} U(1) &= \frac{2}{\gamma+1} \\ Z(1) &= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \end{aligned} \tag{52}$$

(exercise).

# 6 Final solution

Since we have first integral of the equations, the number of independent equations is now reduced from three to two. The easiest equation to drop out of the previous set of 3 is the momentum equation and the most convenient form of the equations to derive the final solution is

$$
\begin{aligned}
-(1-U)\frac{d\ln G}{d\ln\xi} + \frac{dU}{d\ln\xi} + 3U &= 0 && \text{(Continuity)} \\
\frac{d\ln Z}{d\ln\xi} - (\gamma-1)\frac{d\ln G}{d\ln\xi} + \frac{2(U-5/2)}{U-1} &= 0 && \text{(Entropy)} \\
Z(\xi) &= \frac{\gamma(\gamma-1)}{2}\frac{U^2(\xi)[1-U(\xi)]}{[\gamma U(\xi)-1]} && \text{(First integral)}
\end{aligned}
\tag{53}
$$

## Plan

- Eliminate $\dfrac{d\ln G}{d\ln\xi}$ from continuity and entropy equations giving a differential equation connecting $U$ and $Z$.
- Use the first integral also connecting $Z$ and $U$ to solve for $\xi$ as a function of $V$.

Elimination of $\dfrac{d\ln G}{d\ln\xi}$ from continuity and entropy equations gives

$$\frac{dU}{d\ln\xi} - \frac{(1-U)}{\gamma-1}\frac{d\ln Z}{d\ln\xi} - \frac{[5-(3\gamma-1)U]}{\gamma-1} = 0 \qquad (54)$$

Multiply this through by $\dfrac{d\ln\xi}{dU}$ gives

$$1 - \frac{(1-U)}{\gamma-1}\frac{d\ln Z}{dU} - \frac{[5-(3\gamma-1)U]}{\gamma-1}\frac{d\ln\xi}{dU} = 0 \qquad (55)$$

Now we eliminate $d\ln Z/dU$ from this equation using the first integral which can be expressed as:

$$\ln Z = \ln\frac{\gamma(\gamma-1)}{2} + \ln(1-U) + 2\ln U - \ln(\gamma U - 1)$$

$$\Rightarrow \frac{d\ln Z}{dU} = \frac{-1}{1-U} + \frac{2}{U} - \frac{\gamma}{\gamma U - 1} \qquad (56)$$

Substitution of this expression into the above yields, after some simplification:

$$\frac{d\ln\xi}{dU} = \frac{\gamma}{5-(3\gamma-1)U} - \frac{2(1-U)}{U[5-(3\gamma-1)U]} + \frac{\gamma(1-U)}{(\gamma U-1)[5-(3\gamma-1)U]} \tag{57}$$

All that remains to do at this stage is to expand the factors and integrate giving

$$5\ln\xi = -\frac{(13\gamma^2-7\gamma+12)}{(2\gamma+1)(3\gamma-1)}\ln[5-(3\gamma-1)U] - 2\ln U + \frac{5(\gamma-1)}{2\gamma+1}\ln[\gamma U-1] + \text{constant} \tag{58}$$

The constant is determined by the condition

$$U(1) = \frac{2}{\gamma + 1} \tag{59}$$

The result is:

$$\xi^5 = \left[\frac{\gamma+1}{2}U\right]^{-2}\left[\frac{\gamma+1}{7-\gamma}[5-(3\gamma-1)U]\right]^{\nu_1}\left[\frac{\gamma+1}{\gamma-1}(\gamma U-1)\right]^{\nu_2}$$

$$\nu_1 = -\frac{(13\gamma^2 - 7\gamma + 12)}{(2\gamma+1)(3\gamma-1)} \tag{60}$$

$$\nu_2 = \frac{5(\gamma-1)}{2\gamma+1}$$

## Solution for G

Using the continuity equation:

$$-(1-U)\frac{d\ln G}{d\ln\xi} + \frac{dU}{d\ln\xi} + 3U = 0$$

$$\Rightarrow \frac{d\ln G}{dU} = \frac{1}{1-U} + \frac{3U}{1-U}\frac{d\ln\xi}{dU} \qquad (61)$$

We can express $\frac{d\ln\xi}{dU}$ in terms of U using eqn. (57) and then integrate to obtain $G(U(\xi))$. The result is:

$$G(\xi) = \frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{\gamma-1}(\gamma U-1)\right]^{\nu_3}\left[\frac{\gamma+1}{7-\gamma}[5-(3\gamma-1)U]\right]^{\nu_4} \times \left[\frac{\gamma+1}{\gamma-1}(1-U)\right]^{\nu_5} \tag{62}$$

where

$$\nu_3 = \frac{3}{2\gamma + 1}$$

$$\nu_4 = -\frac{\nu_1}{2 - \gamma} = \frac{13\gamma^2 - 2\gamma + 12}{(3\gamma - 1)(2 - \gamma)(2\gamma + 1)} \qquad (63)$$

$$\nu_5 = -\frac{2}{2 - \gamma}$$

## 7 Determination of $\beta$

Change the variable of integration of the integral for $\beta$ from $\xi$ to $U$. Note that $\xi = 0$ corresponds to $U = 1/\gamma$

$$
\begin{aligned}
\beta^5 &= \frac{16\pi}{25}\int_0^1 \xi^4 G(\xi)\left[\frac{1}{2}U^2(\xi) + \frac{1}{\gamma(\gamma-1)}Z(\xi)\right]d\xi \\
&= \frac{16\pi}{25}\int_{1/\gamma}^{2/(\gamma+1)} \xi^5 G(U)\left[\frac{1}{2}U^2 + \frac{1}{\gamma(\gamma-1)}Z(U)\right]\frac{d\ln\xi}{dU}dU
\end{aligned}
\tag{64}
$$

The last integral can be easily integrated numerically given all of the previous expressions for $\xi^5(U)$, $\dfrac{d\ln\xi}{dU}$ etc. (One point to note is that the integral contains a removable singularity at $U = 1/\gamma$)

Typical results are

$$\gamma = 1.4 \Rightarrow \beta = 0.968$$
$$\gamma = \frac{5}{3} \Rightarrow \beta = 0.868 \qquad (65)$$

## 8 Application of Sedov solution to supernova remnants

Take relationship between $r$, $\beta$, $\xi$

$$r = \beta \left( \frac{E}{\rho_0} \right)^{1/5} \xi t^{2/5} \qquad (66)$$

The locus of the outgoing blast wave is given by $\xi = 1$

Knowing the radius of the blast wave and the time we can calculate the energy released by the supernova.

$$E = \rho_0 \left[ \frac{r_{BW}}{\beta t^{2/5}} \right]^5 \tag{67}$$

$$= \rho_0 \beta^{-5} r_{BW}^5 t^{-2}$$

Take ISM number density $n \approx 1\ \mathrm{cm}^{-3} \approx 10^6\ \mathrm{m}^{-3}$

$$E \approx \mu n m \beta^{-5} r_{BW}^5 t^{-2}$$

$$= 5.8 \times 10^{42}\ \mathrm{J} \times \left( \frac{n}{10^6} \right) \left( \frac{r_{BW}}{\mathrm{pc}} \right)^5 \left( \frac{t}{100\ \mathrm{yrs}} \right)^{-2} \tag{68}$$

e.g. Crab nebula

$$r_{BW} = 3\text{pc} \qquad t = (1992 - 1054) = 938 \text{ yrs}$$

$$\Rightarrow E \approx 1.6 \times 10^{43} \text{ J} \tag{69}$$

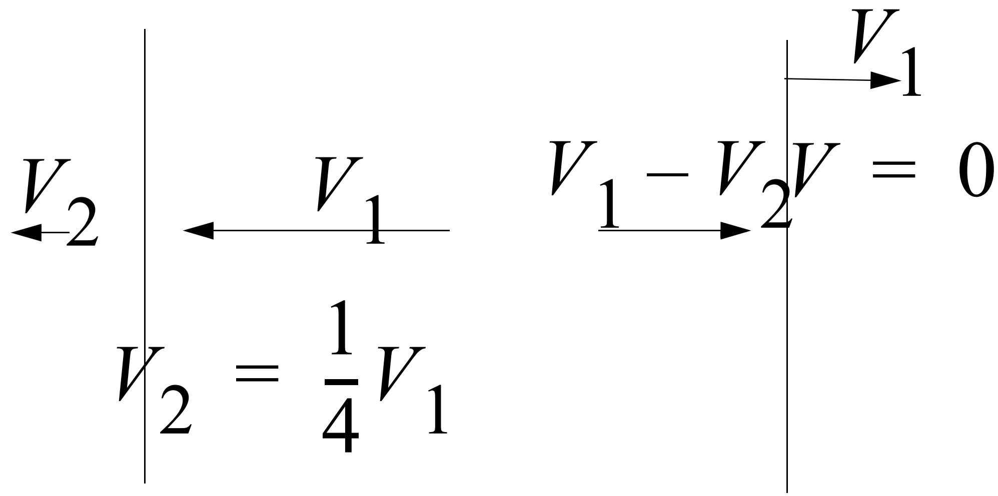
## *Velocity of expansion*

$$\frac{dr}{dt} = \frac{2}{5}\frac{r}{t}$$

$$V_{\text{sh}} = 3900 \text{ km/s}\left(\frac{r}{\text{pc}}\right)\left(\frac{t}{100 \text{ yrs}}\right)^{-1} \tag{70}$$

$$= 1250 \text{ km/s for Crab}$$

# Velocity of shocked gas




For a strong shock $V_1 - V_2 = \frac{3}{4} V_1 = 940$ km/s for Crab. This velocity is typical of the expansion velocity of the Crab nebula filaments.