Exercises in Astrophysical Gas Dynamics
Solutions to Problems 11-12

11. Topology of isothermal stellar wind.

For an isothermal stellar wind the sound speed, a, is constant. The parametric form of the above equations is:

$$\begin{aligned}\frac{dr}{du} &= r^2(v^2 - a^2) = f_1(r, v)\\ \frac{dv}{du} &= v(2a^2r - GM) = f_2(r, v)\end{aligned}$$

As in the case of a variable sound speed the critical point (denoted by subscript 0) is located at

$$v_0 = a \qquad r_0 = \frac{G\,M}{2a^2}$$

Expanding in the neighbourhood of the critical point,

$$r = r_0 + r' \qquad v = v_0 + v'$$

giving the linear equations:

$$\frac{d}{du}\begin{bmatrix} r' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial v} \end{bmatrix}\begin{bmatrix} r' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 2r^2v \\ 2a^2v & 0 \end{bmatrix}\begin{bmatrix} r' \\ v' \end{bmatrix}$$

The corresponding eigenvalue equation is (inserting the critical point condition $v = a$):

$$\begin{vmatrix} \lambda & -2ar^2 \\ -2a^3 & \lambda \end{vmatrix} = 0$$

which gives the eigenvalues:

$$\lambda_{1,2} = \pm 2a^2 r = \pm GM$$

The solution of the linear equations is

$$= A\,\mathbf{v}_1\, \exp\left[-GMu\right] + B\,\mathbf{v_2}\, \exp\left[GMu\right]$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of the above matrix and A and B are constants. The eigenvectors are given by:

$$\mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ \pm\frac{\lambda}{2ar^2} \end{bmatrix} = \begin{bmatrix} 1 \\ \pm\frac{a}{r} \end{bmatrix}$$

Thus the values of the 2 curves through the critical point are:

$$\frac{dv}{dr} = \frac{u_2}{u_1} = \pm\left(\frac{a}{r}\right) = \pm\frac{2a^3}{GM}$$

12. *Numerical solution for isothermal wind.* For a numerical solution it is necessary to express the equations in non-dimensional form. We adopt a scaling to non-dimensional variables, R and V so that the critical point is at $(1, 1)$. Hence:

$$r = r_{\rm cp} R \qquad v = v_{\rm cp}\, V = aV$$

In terms of these variables the hydrodynamical equations become:

$$\begin{aligned} \frac{dR}{du} &= \frac{1}{2} GM\, R^2 (V^2 - 1) \\ \frac{dV}{du} &= GM\, V(R-1) \end{aligned}$$

We can complete the process by introducing a new parameter, u', defined by $du' = GM/2\, du$ so that

$$\begin{aligned} \frac{dR}{du'} &= R^2 (V^2 - 1) \\ \frac{dV}{du'} &= 2V(R-1) \end{aligned}$$

One can redo the eigenvalue analysis for this set of equations or simply rescale what we had in the previous question. Whatever method is used, the solutions in the neighbourhood of the critical point are given by (now dropping the primes on u'):

$$\begin{bmatrix} R-1 \\ V-1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^u + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-u}$$

This equation can be used to start the solution *near* the critical point. By taking initial values corresponding to $C_2 = 0$ and $C_1\, e^u = \pm\epsilon$ where ϵ is small, one is selecting starting points for one of the critical curves on either side of the critical point. Another way of saying this is to note that for $C_2 = 0$, $(R-1)/(V-1) = 1 \Rightarrow (V-1) = (R-1)$. Therefore, by selecting initial values corresponding to $R - 1 = \pm\epsilon \Rightarrow (V-1) = \pm\epsilon$ one selects starting points on either side of the critical point.

Similarly, for $C_1 = 0$, $V - 1 = -(R-1)$ and selecting $R - 1 = \pm\epsilon$ gives $(V-1) = \mp\epsilon$, providing starting points for the other critical curve. In both cases, one integrates away from these points towards $u = infinity$ on both sides. A convenient starting value for u is zero, in each case. The perturbation solution shows that it is the initial values of R and V, which determine the direction along the curve that the solution traverses.

Other curves in the (R, V) plane can be mapped out by selecting reasonable initial values, e.g. $(R-1, V-1) = (1/2, 0)$ and then integrating in each direction away from this point, again starting from $u = 0$, but integrating towards $u = \infty$ in one direction and $u = -\infty$ in the other.

Figure **??** shows the curves of velocity and density through the critical point and 4 other curves which do not traverse the critical point.

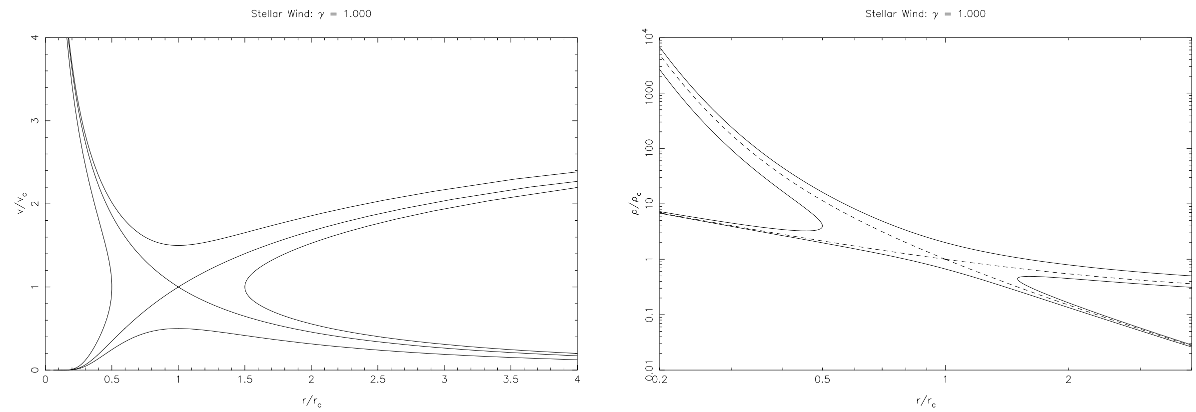

Fig. 1.— Velocity and density for an isothermal wind