Magnetohydrodynamics

MHD

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General physical references

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Maxwell's equations (cgs Gaussian units)

Gauss's law of electrostatics

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e$$

$$\nabla \cdot \mathbf{B} = 0$$

No magnetic monopoles

Ampere's law

Electric current

Displacement current

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_e + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

Faraday's law of induction

Particle equations of motion

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{\mathbf{c}} \times \mathbf{B}\right) - m\nabla\phi$$

Lorentz force Gravitational force

Cartesian form of Maxwell's equations

$$
\begin{aligned}
\frac{\partial E_j}{\partial x_j} &= 4\pi\rho_e \\
\frac{\partial B_j}{\partial x_j} &= 0 \\
\epsilon_{ijk}\frac{\partial B_k}{\partial x_j} &= \frac{4\pi}{c}J_i + \frac{1}{c}\frac{\partial E_i}{\partial t} \\
\epsilon_{ijk}\frac{\partial E_k}{\partial x_j} + \frac{1}{c}\frac{\partial B_i}{\partial t} &= 0
\end{aligned}
$$

Equations of motion of a charged particle

$$
m\frac{dv_i}{dt} = q\left(E_i + \epsilon_{ijk}\frac{v_j}{c}B_k\right) - m\frac{\partial \phi}{\partial x_i}
$$

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Energy density, Poynting flux and Maxwell stress tensor

$$
\begin{aligned}
\epsilon_{\mathrm{EM}} &= \frac{E^2+B^2}{8\pi} = \text{Electromagnetic energy density} \\
S_i &= \frac{c}{4\pi}\epsilon_{ijk}E_jB_k = \text{Poynting flux} \\
\Pi_i^{\mathrm{EM}} &= \frac{S_i}{c^2} = \text{Electromagnetic momentum density} \\
M_{ij}^{\mathrm{B}} &= \frac{1}{4\pi}\left(B_iB_j - \frac{1}{2}B^2\delta_{ij}\right) = \text{Magnetic component of} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{Maxwell stress tensor} \\
M_{ij}^{\mathrm{E}} &= \frac{1}{4\pi}\left(E_iE_j - \frac{1}{2}E^2\delta_{\mathrm{ij}}\right) = \text{Electric component of} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{Maxwell stress tensor}
\end{aligned}
$$

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Relationships between electromagnetic energy, flux and momentum

The following relations can be derived from Maxwell's equations:

$$\begin{aligned} \frac{\partial \epsilon_{\mathrm{EM}}}{\partial t} + \frac{\partial S_i}{\partial x_i} &= -J_i E_i \\ \frac{\partial \Pi_i}{\partial t} - \frac{\partial M_{ij}}{\partial x_j} &= -\left(\rho_e E_i + \epsilon_{ijk} \frac{J_j}{c} B_k \right) \end{aligned}$$

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Momentum equations

Consider the electromagnetic force acting on a particle

$$F_i^\alpha = q^\alpha \left[E_i + \epsilon_{ijk} \frac{v_j^\alpha}{c} B_k \right]$$

α refers to specific particle

Consider a unit volume of gas and the electromagnetic force acting on this volume

$$F_i^{\mathrm{em}} = \left[\sum_\alpha q^\alpha \right] E_i + \epsilon_{ijk} \left[\sum_\alpha q^\alpha \frac{v_j^\alpha}{c} \right] B_k$$

where the sum is over all particles within the unit volume. N.B. The velocity here is the particle velocity not the fluid velocity.

Momentum (cont'd)

$$F_i^{\mathrm{em}} = \left[\sum_\alpha q^\alpha \right] E_i + \epsilon_{ijk} \left[\sum_\alpha q^\alpha v_j^\alpha \right] B_k$$

We can identify the following components

$$\sum_\alpha q^\alpha \quad = \quad \rho_e = \text{Electric charge density}$$

$$\sum_\alpha q^\alpha v_j^\alpha \quad = \quad J_i = \text{Electric current density}$$

so that the electromagnetic force can be written

$$F_i^{\mathrm{em}} = \rho_e \, E_i + \epsilon_{ijk} \frac{J_j}{c} B_k$$

$$\text{i.e.} \quad \mathbf{F} = \rho_e \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}$$

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Momentum (cont'd)

We add the body force to the momentum equations to obtain

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \rho_e E_i + \epsilon_{ijk} \frac{J_j}{c} B_k$$

Now use the equation for the conservation of electromagnetic momentum:

$$\rho_e E_i + \frac{1}{c} \epsilon_{ijk} J_j B_k = -\frac{\partial \Pi_i}{\partial t} + \frac{\partial M_{ij}}{\partial x_j}$$

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Momentum (cont'd)

so that the momentum equations become:

Bulk transport of momentum

Flux of momentum due to EM field

$$\frac{\partial}{\partial t}(\rho v_i + \Pi_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p\delta_{ij} - M_{ij}) = -\rho \frac{\partial \phi}{\partial x_i}$$

Mechanical + EM momentum density

Flux of momentum due to pressure

Gravitational force

For non-relativistic motions and large conductivity some very useful approximations are possible

Limit of infinite conductivity

In the plasma rest frame (denoted by primes), Ohm's law is

$$J_i' = \sigma E_i'$$

Conductivity

The conductivity of a plasma is very high so that for a finite current

$$E_i' \approx 0$$

This has implications for the lab-frame electric and magnetic field

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Transformation of electric and magnetic fields

Lorentz transformation of electric and magnetic fields

$$\begin{aligned} \mathbf{E}' &= \Gamma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \\ \mathbf{B}' &= \Gamma \left(\mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c} \right) \end{aligned}$$

$$\Gamma = \text{Lorentz factor}$$

$$\text{If} \quad \mathbf{E}' = \mathbf{0}$$

$$\begin{aligned} \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} &= 0 \\ \Rightarrow \mathbf{E} &= -\frac{\mathbf{v} \times \mathbf{B}}{c} = \mathcal{O}\left(\frac{vB}{c} \right) \end{aligned}$$

This is the magnetohydrodynamic approximation.

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Maxwell tensor

Since $$\mathbf{E} = \mathcal{O}\left(\frac{vB}{c}\right)$$

then

$$M_{ij}^{\mathrm{E}} = \mathcal{O}\left(\frac{v^2}{c^2}\right) \times M_{ij}^{\mathrm{B}}$$

Hence, we neglect the electric component of the Maxwell tensor. We can also neglect the displacement current, as we show later.

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Electromagnetic momentum

We want to compare the electromagnetic momentum density with the matter momentum density, i.e. compare

$$\Pi_i^{\mathrm{EM}} = \frac{1}{4\pi c}\epsilon_{ijk}E_j B_k \qquad \text{with} \qquad \rho v_i$$

$$\Pi_i^{\mathrm{EM}} = \mathcal{O}\left(\frac{vB^2}{c^2}\right) \qquad\qquad \rho v_i = \mathcal{O}(\rho v)$$

$$\frac{\Pi_i^{EM}}{\rho v_i} = \mathcal{O}\left(\frac{B^2}{4\pi\rho c^2}\right)$$

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Electromagnetic momentum

$$\frac{\Pi_i^{EM}}{\rho v_i} = \mathcal{O}\left(\frac{B^2}{4\pi\rho c^2}\right)$$

As we shall discover later, the quantity

$$\frac{B^2}{4\pi\rho} = v_A^2 = (\text{Alfven speed})^2$$

where the Alfven speed is a characteristic wave speed within the plasma. We assume that the magnetic field is low enough and/or the density is high enough such that

$$\frac{v_A^2}{c^2} \ll 1$$

and we neglect the electromagnetic momentum density.

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Final form of the momentum equation

Given the above simplifications, the final form of the momentum equations is:

$$\boxed{\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi}\delta_{ij}\right]}$$

We also have

$$\frac{1}{4\pi}\mathrm{curl}\mathbf{B} \times \mathbf{B} = \mathrm{div}\left[\frac{\mathbf{BB}}{4\pi} - \frac{B^2}{8\pi}\mathbf{I}\right]$$

where **I** is the unit tensor

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Thus the momentum equations can be written:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \rho \nabla \phi + \frac{1}{4\pi} \mathrm{curl} \mathbf{B} \times \mathbf{B}$$

Either form can be more useful dependent upon circumstances.

Displacement current

Let L be a characteristic length, T a characteristic time and V = L/T a characteristic velocity in the system

The equation for the current is:

$$
\begin{aligned}
J_i &= \frac{c}{4\pi}\epsilon_{ilm}\frac{\partial B_m}{\partial x_l} - \frac{1}{4\pi}\frac{\partial E_i}{\partial t} \\
& \qquad \mathcal{O}\left(\frac{cB}{L}\right) \qquad \mathcal{O}\left(\frac{E}{T} = \frac{vB}{cT}\right) \\
\Rightarrow \frac{\text{Displacement Current}}{\text{Curl B current}} &= \mathcal{O}\left(\frac{vV}{c^2}\right) \ll 1
\end{aligned}
$$

Displacement current (cont'd)

In the MHD approximation we always put

$$\begin{aligned} J_i &= \frac{c}{4\pi}\epsilon_{ilm}\frac{\partial B_m}{\partial x_l} \\ \text{i.e.} \quad \mathbf{J} &= \frac{c}{4\pi}\mathrm{curl}\,\mathbf{B} \end{aligned}$$

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Energy equation

The total electromagnetic energy density is

$$
\begin{aligned}
E^{\rm EM} &= \frac{E^2 + B^2}{8\pi} \approx \frac{B^2}{8\pi} \\
\text{since} \quad E^2 &= \mathcal{O}\left(\frac{v^2}{c^2}\right) B^2
\end{aligned}
$$

In order to derive the total energy equation for a magnetised gas, we add the electromagnetic energy to the total energy and the Poynting flux to the energy flux

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Energy equation (cont'd)

The final result is:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \epsilon + \rho\phi + \frac{B^2}{8\pi}\right] \quad &+ \quad \frac{\partial}{\partial x_j}\left[\rho\left(\frac{1}{2}v^2 + h + \phi\right)v_i + S_i\right] \\ &= \quad \rho k T\frac{ds}{dt}\end{aligned}}$$

$$\begin{aligned} h \quad &= \quad \frac{\epsilon + p}{\rho} \\ S_i \quad &= \quad \frac{c}{4\pi}\epsilon_{ijk}E_j B_k \\ &= \quad \frac{1}{4\pi}\left[B^2 v_i - B_j v_j B_i\right] = \frac{B^2}{4\pi}v_i^{\perp} \\ \text{where} \quad v_i^{\perp} \quad &= \quad \text{Component of velocity} \\ & \qquad \text{perpendicular to magnetic field}\end{aligned}$$

The induction equation

The final equation to consider is the induction equation, which describes the evolution of the magnetic field.

We have the following two equations:

$$\text{Faraday's Law:} \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{Infinite conductivity:} \qquad \mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B}$$

Together these imply the ***induction equation***

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{v} \times \mathbf{B})}$$

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Summary of MHD equations

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

Momentum:

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi}\delta_{ij}\right]$$

Summary (cont.)

Energy:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \epsilon + \rho\phi + \frac{B^2}{8\pi}\right] = \frac{\partial}{\partial x_j}\left[\left(\frac{1}{2}v^2 + h + \phi\right)\rho v_i + S_i\right]$$
$$= \rho k T \frac{ds}{dt}$$

where

$$h = \frac{\epsilon + p}{\rho} \qquad S_i = \frac{c}{4\pi}\epsilon_{ijk}E_j B_k = \frac{B^2}{4\pi}v_i^{\perp}$$

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Summary (cont.)

Induction:

$$\frac{\partial B_i}{\partial t} = [\mathrm{curl}(\mathbf{v} \times \mathbf{B})]_i$$

$$\frac{\partial B_i}{\partial t} = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} v_l B_m)$$

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