Hydrodynamic Instabilities

References

Chandrasekhar: ***Hydrodynamic and Hydromagnetic Instabilities***

Landau & Lifshitz: ***Fluid Mechanics***

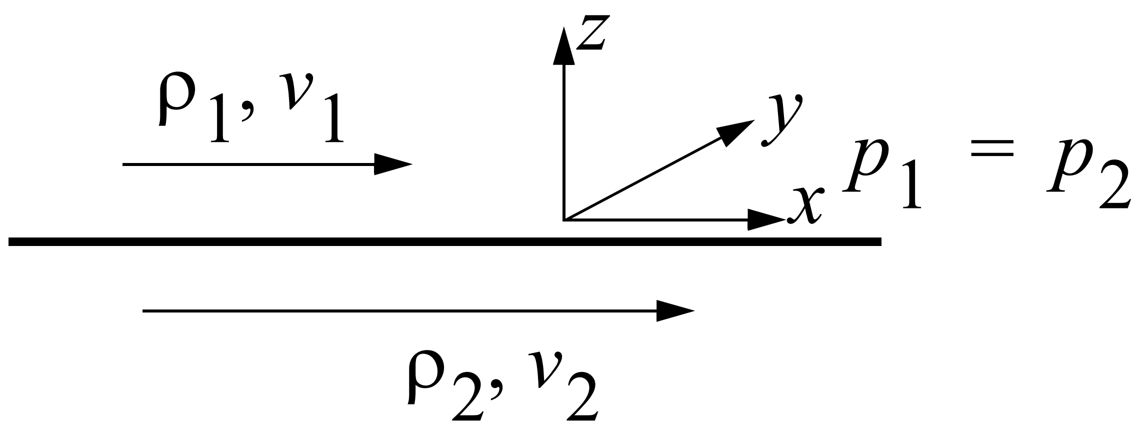
Shu: ***Gas Dynamics***

1 Introduction

Instabilities are an important aspect of any dynamical system. It is one thing to establish the dynamical equations for some system or other, it is another to establish that the system is stable. If it is unstable, then the system will evolve to some other state. For example, we showed, in dealing with shocks that there are two types of discontinuities - the shock discontinuity in which there is a mass flux across the discontinuity and the tangential contact discontinuity for which the mass flux is zero. The latter discontinuity is subject to a classical discontinuity - the Kelvin-Helmholtz discontinuity which is one of the subjects of this chapter.

2 The incompressible Kelvin-Helmholtz instability

The Kelvin-Helmholtz instability relates to the following situation.



A large number of the properties of this instability can be understood in terms of the following incompressible analysis.

Continuity equation for incompressible flow

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{v}) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho + \rho \,\mathrm{div}\, \boldsymbol{v} = 0 \qquad (1)$$

$$\Rightarrow \frac{d\rho}{dt} + \rho \,\mathrm{div}\, \boldsymbol{v} = 0$$

Incompressibility means that the density is constant along a streamline, so that $\frac{d\rho}{dt} = 0$ and

$$\mathrm{div}\ \boldsymbol{v} = \mathbf{0} \qquad (2)$$

Perturbation

To study stability, we perturb the above system as follows:

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}_0 + \boldsymbol{v}' \\ p &= p_0 + p' \end{aligned} \tag{3}$$

where the 0 subscripts refer to the unperturbed flow. Thus

$$\begin{aligned} \boldsymbol{v}_0 &= (v_1, 0, 0) \qquad z > 0 \\ \boldsymbol{v}_0 &= (v_2, 0, 0) \qquad z < 0 \end{aligned} \tag{4}$$

In developing the equations which describe the development of the instability, we develop equations which refer to either $z < 0$ or $z > 0$ as far as possible, introducing either v_1 or v_2 when necessary, when we come to consider the boundary conditions at the interface.

Perturbation of the continuity equation

This is simply

$$\mathrm{div}\,\boldsymbol{v}' = 0 \tag{5}$$

Perturbation of the momentum equation

The term

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} = (\boldsymbol{v}_0 + \boldsymbol{v}')\cdot\nabla\boldsymbol{v}' = \boldsymbol{v}_0\cdot\nabla\boldsymbol{v}' \tag{6}$$

to first order, so that the perturbed momentum equation is:

$$\rho\left[\frac{\partial}{\partial t}\boldsymbol{v}' + \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}'\right] = \rho\left[\frac{\partial}{\partial t}\boldsymbol{v}' + v_{0,\,x}\frac{\partial}{\partial x}\boldsymbol{v}'\right] = -\nabla p' \qquad (7)$$

Take the divergence of this equation

$$\Rightarrow \rho\left[\frac{\partial}{\partial t}\mathrm{div}\boldsymbol{v} + v_{0,\,x}\frac{\partial}{\partial x}\mathrm{div}\boldsymbol{v}'\right] = -\nabla^2 p' \qquad (8)$$

Since $\mathrm{div}\boldsymbol{v}' = 0$,

$$\nabla^2 p' = 0 \qquad (9)$$

Form of the perturbation

Take all perturbed quantities to be of the form:

$$f(\mathbf{r}, t) = f^0(z) \exp i[k_x x + k_y y - \omega t] \tag{10}$$

where k_x and k_y are real components of the wave vector. (Note the use of a 0 *superscript* to indicate the amplitude, as distinct from the 0 *subscript* which characterizes the unperturbed initial state.

Perturbation equation for the pressure

Take

$$p' = p^0 \exp i[k_x x + k_y y - \omega t]$$

$$\Rightarrow \nabla^2 p' = \left[\frac{d^2}{dz^2}p^0 - (k_x^2 + k_y^2)p^0\right]\exp i[k_x x + k_y y - \omega t] = 0 \qquad (11)$$

The magnitude of the wave vector is given by

$$k^2 = k_x^2 + k_y^2 \qquad (12)$$

Hence

$$\frac{d^2}{dz^2}p^0 + k^2 p^0 = 0 \Rightarrow p^0(z) = A_1 e^{kz} + A_2 e^{-kz} \qquad (13)$$

We take different parts of this solution on different sides of the interface. Since the pressure is finite at $z = \pm\infty$ then

$$\begin{aligned} p^0 &= A_1 e^{kz} \qquad z < 0 \\ p^0 &= A_2 e^{-kz} \qquad z > 0 \end{aligned} \qquad (14)$$

We now impose the boundary condition that the perturbed pressures on both sides of the interface are equal so that

$$A_1 = A_2 = A \qquad (15)$$

and

$$
\begin{aligned}
p^0 &= Ae^{kz} \qquad && z < 0 \\
p^0 &= Ae^{-kz} \qquad && z > 0
\end{aligned}
\tag{16}
$$

Perturbation equation for the velocity

Taking,

$$v_z{}' = v_z^0(z)\exp i[k_x x + k_y y - \omega t]$$

$$\Rightarrow \frac{\partial}{\partial x} v_z{}' = v_z^0(z) \times i k_x \exp i[k_x x + k_y y - \omega t]$$

$$\frac{\partial v_z{}'}{\partial x} = v_z^0(z) \times (i k_x)\exp i[k_x x + k_y y - \omega t] \tag{17}$$

$$\frac{\partial v_z{}'}{\partial t} = v_z^0(z) \times (-i\omega)\exp i[k_x x + k_y y - \omega t]$$

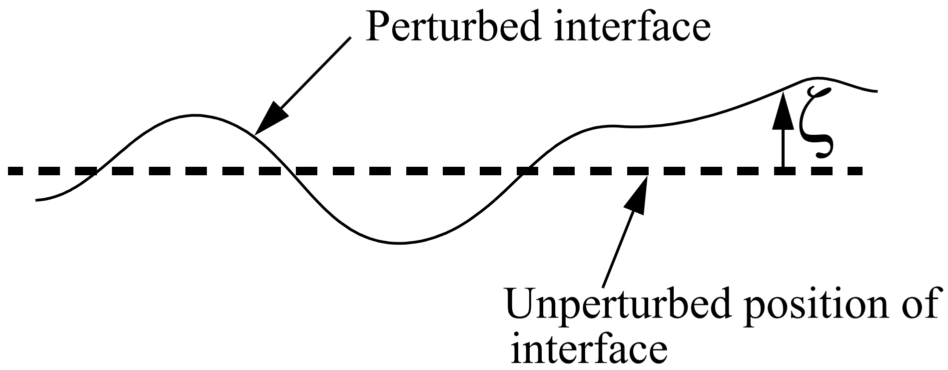
We substitute these expressions into the momentum equation to obtain:

$$\rho_0[v_z^0(z)(k_x v_0 - \omega)] \times \exp i[k_x x + k_y y - \omega t] = -\frac{\partial p'}{\partial z}$$
$$= \pm A e^{\mp kz} \exp i[k_x x + k_y y - \omega t] \qquad (18)$$

where the different signs refer to $z > 0$ and $z < 0$ respectively. This gives us the following solution for $v_z^0(z)$:

$$v_z^0(z) = \pm \frac{A e^{\mp kz}}{\rho_0(k_x v_0 - \omega)} \qquad (19)$$

Displacement of the surface



To proceed further, we need to consider the displacement of the fluid at the interface.

Consider the displacement of a fluid element at any position in the fluid. A given fluid element satisfies:

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \tag{20}$$

so that putting

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{r}' \tag{21}$$

where $\boldsymbol{r}'$ is the variation from the zeroth order flow as a result of the perturbation, gives

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{r}_0 + \boldsymbol{r}') &= \boldsymbol{v}_0 + \boldsymbol{v}' \\ \Rightarrow \frac{d\boldsymbol{r}'}{dt} &= \boldsymbol{v}' \end{aligned} \tag{22}$$

The differentiation on the left hand side is "following the motion" so that this perturbation equation is, in fact,

$$\frac{\partial \boldsymbol{r}'}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{r}' = \frac{\partial \boldsymbol{r}'}{\partial t} + v_{0,x} \frac{\partial}{\partial x} \boldsymbol{r}' = \boldsymbol{v}' \tag{23}$$

The component of this set of equation which is of the most use to us, is the z component. Denoting the z component of $\boldsymbol{r}'$ by ζ,

$$\frac{\partial \zeta}{\partial t} + v_{0,x}\frac{\partial \zeta}{\partial x} = v_z' = \pm\frac{Ae^{\mp kz}}{\rho_0(k_x v_0 - \omega)} \qquad (24)$$

As with all other functions, we put,

$$\zeta = \zeta^0 \exp i[k_x x + k_y y - \omega t]$$

$$\frac{\partial \zeta}{\partial t} = -i\omega\zeta^0 \exp i[k_x x + k_y y - \omega t] \qquad (25)$$

$$\frac{\partial \zeta}{\partial x} = ik_x\zeta^0 \exp i[k_x x + k_y y - \omega t]$$

Therefore, the equation for ζ becomes:

$$i\zeta^0(k_x v_0 - \omega) \times \exp i[k_x x + k_y y - \omega t] = \pm\frac{Ae^{\mp kz}}{\rho_0(k_x V_0 - \omega)} \times \exp i[k_x x + k_y y - \omega t] \qquad (26)$$

$$\Rightarrow \zeta^0 = \pm\frac{Ae^{\mp kz}}{\rho_0(k_x v_0 - \omega)^2}$$

Now, at $z = 0$, the displacements calculated from either side of the interface should be identical. Therefore,

$$\rho_1(k_x v_1 - \omega)^2 = -\rho_2(k_x v_2 - \omega)^2$$

$$\Rightarrow \rho_1(k_x v_1 - \omega)^2 + \rho_2(k_x v_2 - \omega)^2 = 0 \qquad (27)$$

Expanding out the quadratic terms:

$$\rho_1[\omega^2 - 2\omega k_x v_1 + k_x^2 v_1^2] + \rho_2[\omega^2 - 2\omega k_x v_2 + k_x^2 v_2^2] = 0$$

$$(\rho_1 + \rho_2)\omega^2 - 2\omega k_x[\rho_1 v_1 + \rho_2 v_2] + k_x^2[\rho_1 v_1^2 + \rho_2 v_2^2] = 0 \tag{28}$$

$$\Rightarrow (\rho_1 + \rho_2)\left(\frac{\omega}{k_x}\right)^2 - 2\left(\frac{\omega}{k_x}\right)[\rho_1 v_1 + \rho_2 v_2] + [\rho_1 v_1^2 + \rho_2 v_2^2] = 0$$

and the solution of the quadratic equation for ω / k_x is

$$\frac{\omega}{k_x} = \frac{(\rho_1 v_1 + \rho_2 v_2) \pm i(v_1 - v_2)(\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2} \tag{29}$$

This is our main result, the dispersion relationship between frequency and wave number. The important feature of this solution is that it has both real and imaginary parts:

$$\frac{\omega_R}{k_x} = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2} \qquad \frac{\omega_I}{k_x} = (v_1 - v_2)\frac{(\rho_1\rho_2)^{1/2}}{\rho_1 + \rho_2} \tag{30}$$

The imaginary part corresponds to both exponentially decaying and growing solutions, since

$$\exp i[\omega_R \pm i\omega_I t] = \exp i\omega_R \times \exp \mp \omega_I t \tag{31}$$

An arbitrary set of initial conditions will give both decaying and growing solutions, so that the above solution enables us to identify a growth rate,

$$\omega_g = (v_1 - v_2)\frac{(\rho_1\rho_2)^{1/2}}{\rho_1 + \rho_2}k_x = \frac{\eta^{1/2}}{1+\eta}(v_1 - v_2)k_x \tag{32}$$

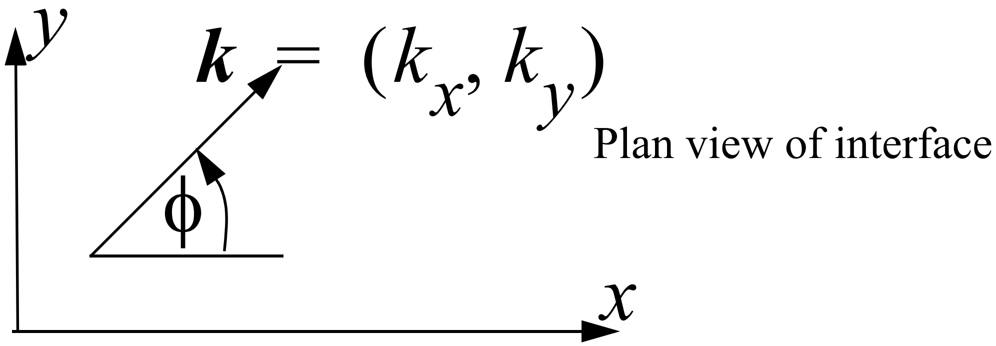
where, the density ratio,

$$\eta = \frac{\rho_1}{\rho_2} \tag{33}$$

and

$$k_x = k\cos\phi \tag{34}$$

is the component of the wave number in the direction of flow.



Features

- Growth depends upon there being a velocity difference.
- Growth rate proportional to k_x (component of wave number in the direction of flow) so that the smallest waves (largest k_x) grow the fastest.
- The growth rate reduces to zero for waves perpendicular to

the direction of motion.

- The growth rate is a maximum for $\eta = 1$.
- All perturbations diminish exponentially away from the interface. (Perturbation $\propto \exp \pm kz$) This is a characteristic feature of *surface waves*.
- These features are also characteristic of the KH instability for compressible flows (as we show in the next section).

3 Compressible Kelvin-Helmholtz instability

3.1 General comments

When we deal with compressible flow, the main complicating factor is that we have to deal with are sound waves when we perturb the flow. In one sense we can think of the KH instability as sound waves in an inhomogeneous medium. (Sound waves are emitted as the surface is disturbed.)

The consideration of the compressible KH instability is similar to that of the incompressible KH instability with some complications related to compressibility. The steps in the development are:

- Determine the dispersion relations for waves in each medium.
- Apply boundary conditions at $z = \pm\infty$ and at the interface.

This leads to polynomial equations and conditions on the roots of these leads to useful information.

3.2 Dispersion relations in each medium

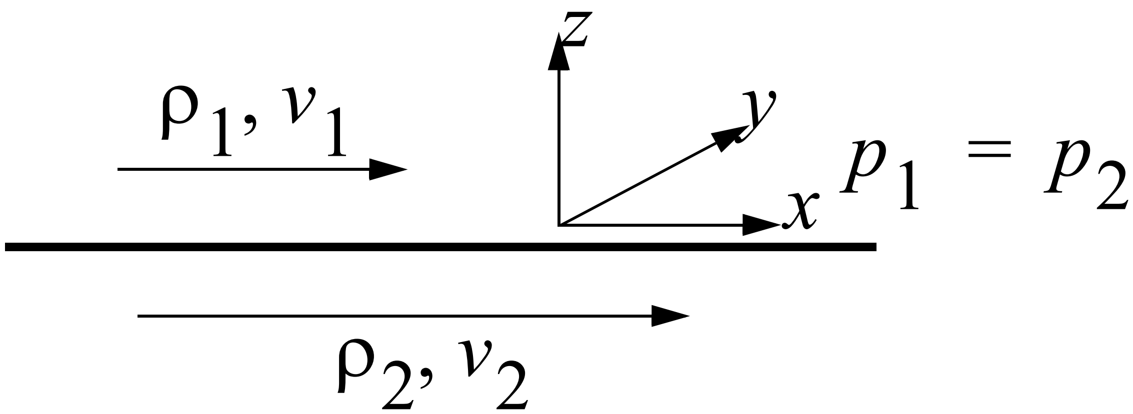
We start with the usual momentum and continuity equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] + \frac{\partial p}{\partial x_i} = 0 \qquad (35)$$

and perturb them according them according to the usual recipe:

$$\rho = \rho_0 + \rho' \qquad v_i = v_{0,\,i} + v_i' \qquad (36)$$



These perturbations imply that

$$\rho v_i = \rho_0 v_{0,i} + \rho_0 v_i' + \rho' v_{0,i} \tag{37}$$

(used in the continuity equation) and

$$v_j \frac{\partial v_i}{\partial x_j} = (v_{0,j} + v_j') \frac{\partial}{\partial x_j} v_i' = v_0 \frac{\partial}{\partial x} v_i' \tag{38}$$

(used in the momentum equation).

Make these substitutions

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_j'}{\partial x_j} + v_{0,j} \frac{\partial \rho'}{\partial x_j} = 0$$

$$\rho_0 \left[\frac{\partial v_i}{\partial t} + v_0 \frac{\partial v_i'}{\partial x} \right] + \frac{\partial p'}{\partial x_i} = 0 \tag{39}$$

In the following, we adopt the pressure as the primary variable since it is continuous across the interface rather than the density which may be discontinuous. In doing so we use,

$$\left\{\begin{array}{c}\dfrac{\partial}{\partial t}\\[2ex]\dfrac{\partial}{\partial x_i}\end{array}\right\}\rho = \frac{1}{c_s^2}\left\{\begin{array}{c}\dfrac{\partial}{\partial t}\\[2ex]\dfrac{\partial}{\partial x_i}\end{array}\right\}p \qquad (40)$$

Hence, the perturbed continuity equation becomes

$$\frac{1}{c_0^2}\left[\frac{\partial p'}{\partial t} + v_0\frac{\partial p'}{\partial x}\right] + \rho_0\frac{\partial v_j'}{\partial x_j} = 0 \qquad (41)$$

Summary of perturbation equations

Combine ρ_0 and c_0^2 in continuity equation and put together with momentum equation:

$$\left[\frac{\partial p'}{\partial t} + v_0 \frac{\partial p'}{\partial x}\right] + \rho_0 c_0^2 \frac{\partial v_j'}{\partial x_j} = 0 \tag{42}$$

$$\rho_0 \left[\frac{\partial v_i}{\partial t} + v_0 \frac{\partial v_i'}{\partial x}\right] + \frac{\partial p'}{\partial x_i} = 0 \tag{43}$$

Similar to the compressible case, we take perturbations of p' and v_i' proportional to

$$\exp i[k_x x + k_y y + k_z z - \omega t] \qquad (44)$$

where (k_x, k_y) are real and k_z and ω may be complex. Compare this with the z-dependence for the incompressible Kelvin-Helmholtz instability $\propto e^{\pm kz}$.

Take

$$\begin{aligned} p' &= A \exp i[k_x x + k_y y + k_z z - \omega t] \\ v_i' &= A_i \exp i[k_x x + k_y y + k_z z - \omega t] \end{aligned} \qquad (45)$$

With these dependencies:

$$\left[\frac{\partial p'}{\partial t} + v_0\frac{\partial p'}{\partial x}\right] = -i(\omega - k_x v_0)A\exp i[k_x x + k_y y + k_z z - \omega t]$$

$$\left[\frac{\partial v_i'}{\partial t} + v_0\frac{\partial v_i'}{\partial x}\right] = -i(\omega - k_x v_0)A_i\exp i[k_x x + k_y y + k_z z - \omega t]$$

$$\frac{\partial v_j'}{\partial x_j} = ik_j A_j\exp i[k_x x + k_y y + k_z z - \omega t]$$

Putting all of this together, using $\rho c_0^2 = \gamma p_0$,

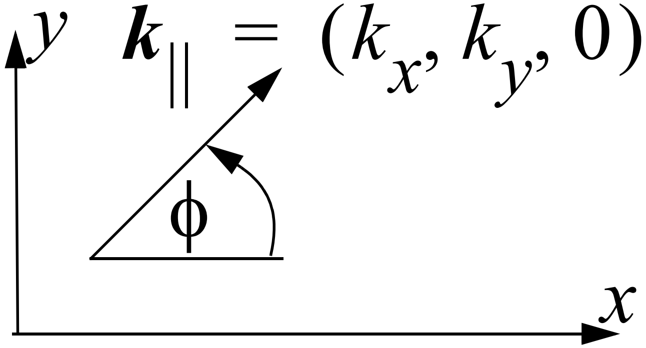
$$\begin{aligned} -i(\omega - k_x v_0)A + \gamma p_0(ik_j A_j) &= 0 \\ -i\rho_0(\omega - k_x V_0)A_i + ik_i A &= 0 \end{aligned} \qquad (46)$$

Notation for wave vector

At this point we introduce the following notation for the wave vector

$$
\begin{aligned}
\boldsymbol{k} &= (k_x, k_y, k_z) = (\boldsymbol{k}_{\|}, k_z) \\
k_x &= k_{\|}\cos\phi \qquad k_y = k_{\|}\sin\phi \\
k^2 &= k_{\|}^2 + k_z^2 = k_i k_i
\end{aligned}
\tag{47}
$$

where $\boldsymbol{k}_{\|}$ is the component of the wave vector parallel to the interface.



$$\boldsymbol{k}_{\parallel} = (k_x, k_y, 0)$$

$$k_x = k_{\parallel}\cos\phi$$

$$k_y = k_{\parallel}\sin\phi$$

Plan view of interface

With this notation the perturbation equations for continuity and momentum become:

$$\begin{aligned} -i(\omega - k_{\parallel}v_0\cos\phi)A + \gamma p^0(ik_jA_j) &= 0 \\ -i(\omega - k_{\parallel}v_0\cos\phi)\rho_0A_i + ik_iA &= 0 \end{aligned} \qquad (48)$$

Take the scalar product of the second of the above equations with k_i. This gives

$$-i(\omega - k_{\|}v_0\cos\phi)\rho_0 k_i A_i + ik^2 A = 0 \qquad (49)$$

$$\Rightarrow ik_i A_i = \frac{ik^2 A}{\rho_0(\omega - k_{\|}v_0\cos\phi)}$$

Substitute this result back into the first of the perturbation equations.

$$(50)$$

$$
\begin{aligned}
-i(\omega - k_{\|} V_0 \cos\phi)A + \gamma p_0 (i k_j A_j) &= -i(\omega - k_{\|} v_0 \cos\phi)A \\
&\quad + \frac{\gamma p_0}{\rho_0} \frac{i k^2 A}{(\omega - k_{\|} v_0 \cos\phi)} \\
&= 0
\end{aligned}
$$

Solving, and dividing out the common factor of A

$$
(\omega - k_{\|} v_0 \cos\phi)^2 = \frac{\gamma p_0}{\rho_0} k^2 = c_0^2 k^2 = c_0^2 (k_{\|}^2 + k_z^2) \qquad (51)
$$

Note that we have essentially recovered the dispersion equation for sound waves!

For the two different sides of the interface,

$$(\omega - k_{\|} v_1 \cos\phi)^2 = c_1^2 k^2$$
$$(\omega - k_{\|} v_2 \cos\phi)^2 = c_2^2 k^2 \qquad (52)$$

Perturbation of the surface

As before consideration of the perturbation of the surface is important.

The z-component of the displacement is the same as for the incompressible case and is given by

$$\frac{\partial \zeta}{\partial t} + v_0 \frac{\partial \zeta}{\partial x} = v_z' = A_z \exp i[k_x x + k_y y + k_z z - \omega t] \qquad (53)$$

Putting

$$\zeta = B_z \exp i[k_x x + k_y y + k_z z - \omega t] \tag{54}$$

gives

$$-i[\omega - k_{||} v_0 \cos\phi] B_z = A_z$$

$$\Rightarrow B_z = \frac{iA_z}{[\omega - k_{||} v_0 \cos\phi]} \tag{55}$$

Now we have from the perturbation equations:

$$-i(\omega - k_{||} v_0 \cos\phi)\rho_0 A_i + ik_i A = 0 \tag{56}$$

so that

$$A_z = \frac{k_z A}{\rho_0(\omega - k_{\|} v_0 \cos\phi)} \tag{57}$$

and therefore, B_z, the coefficient of the z component of displacement is given in terms of A, the coefficient of the pressure field, by

$$B_z = \frac{i k_z A}{\rho_0(\omega - k_{\|} v_0 \cos\phi)^2} \tag{58}$$

This is a generic equation which applies to either region. However, both the displacement, and the pressure are continuous at the interface. Therefore B_z/A is continuous, i.e.

$$\frac{B_{z,1}}{A_1} = \frac{B_{z,2}}{A_2} \tag{59}$$

and

$$\frac{k_{1,z}}{\rho_1(\omega - k_{\|}v_1\cos\phi)^2} = \frac{k_{2,z}}{\rho_2(\omega - k_{\|}v_2\cos\phi)^2} \tag{60}$$

Now return to the dispersion relations which we derived for the two regions, viz,

$$(\omega - k_{\|} v_1 \cos\phi)^2 = c_1^2 k^2 = c_1^2 (k_{\|}^2 + k_{1,z}^2)$$
$$(\omega - k_{\|} v_2 \cos\phi)^2 = c_2^2 k^2 = c_2^2 (k_{\|}^2 + k_{2,z}^2) \qquad (61)$$

We make k_z^2 the subject of these equations,

$$k_{1,z}^2 = \frac{(\omega - k_{\|} v_1 \cos\phi)^2}{c_1^2} - k_{\|}^2$$
$$k_{2,z}^2 = \frac{(\omega - k_{\|} v_2 \cos\phi)^2}{c_2^2} - k_{\|}^2 \qquad (62)$$

In the equation derived from the boundary condition, we put

$$\rho_1 = \frac{\gamma_1 p_0}{c_1^2} \qquad \rho_2 = \frac{\gamma_2 p_0}{c_2^2} \tag{63}$$

giving us

$$\frac{k_{1,z}}{\dfrac{\gamma_1(\omega - k_{\parallel} v_1 \cos\phi)^2}{c_1^2}} = \frac{k_{2,z}}{\dfrac{\gamma_2(\omega - k_{\parallel} v_2 \cos\phi)^2}{c_2^2}} \tag{64}$$

Squaring,

$$\frac{k_{1,z}^2}{\dfrac{\gamma_1^2(\omega - k_{\|} V_1 \cos\phi)^4}{c_1^4}} = \frac{k_{2,z}^2}{\dfrac{\gamma_2^2(\omega - k_{\|} V_2 \cos\phi)^4}{c_2^4}} \tag{65}$$

and substituting for k_z^2

$$\frac{\dfrac{(\omega - k_{\|} v_1 \cos\phi)^2}{c_1^2} - k_{\|}^2}{\dfrac{\gamma_1^2(\omega - k_{\|} v_1 \cos\phi)^4}{c_1^4}} = \frac{\dfrac{(\omega - k_{\|} v_2 \cos\phi)^2}{c_2^2} - k_{\|}^2}{\dfrac{\gamma_2^2(\omega - k_{\|} v_2 \cos\phi)^4}{c_2^4}} \tag{66}$$

This can actually be simplified! We divide the numerators by $k_{\|}^2$ and the denominators by $k_{\|}^4$ and put $v_{\mathrm{ph}} = \dfrac{\omega}{k_{\|}}$, the phase velocity of the wave. This gives,

$$\frac{\dfrac{(v_{\mathrm{ph}} - v_1\cos\phi)^2}{c_1^2} - 1}{\dfrac{\gamma_1^2(v_{\mathrm{ph}} - v_1\cos\phi)^4}{c_1^4}} = \frac{\dfrac{(v_{\mathrm{ph}} - v_2\cos\phi)^2}{c_2^2} - 1}{\dfrac{\gamma_2^2(v_{\mathrm{ph}} - v_2\cos\phi)^4}{c_2^4}} \qquad (67)$$

Furthermore, we can make a Galilean transformation in the x direction in which the velocity, V_1, of the lower stream is zero, i.e.

$$x' = x - v_1 t \tag{68}$$

and this transforms $k_x x - \omega t$ to

$$[k_x(x' + v_1 t) - \omega t] = [k_x x' - (\omega - k_x v_1)t] \tag{69}$$

Therefore, in the new frame,

$$\omega' = \omega - k_x v_1 \Rightarrow \omega = \omega' + k_x v_1 \tag{70}$$

and

$$\begin{aligned}\omega - k_x v_2 \cos\phi &= \omega' - k_x (v_2 - v_1) \cos\phi \\ &= \omega' - k_x \Delta v \cos\phi\end{aligned} \tag{71}$$

where

$$\Delta v = v_2 - v_1 \tag{72}$$

is the difference in velocity between the two streams.

Hence,

$$\begin{aligned} \frac{\omega}{k_{\|}} - v_1 \cos\phi &= \frac{\omega'}{k_{\|}} \\ \frac{\omega}{k_{\|}} - v_2 \cos\phi &= \frac{\omega'}{k_{\|}} - \Delta v \cos\phi \end{aligned} \qquad (73)$$

i.e.

$$\begin{aligned} v_{\mathrm{ph}} - v_1 \cos\phi &= v_{\mathrm{ph}}' \\ v_{\mathrm{ph}} - v_2 \cos\phi &= v_{\mathrm{ph}}' - \Delta v \cos\phi \end{aligned} \qquad (74)$$

where $v_{\mathrm{ph}}' = \dfrac{\omega'}{k_{\|}}$ is the phase velocity in the primed frame, the one in which the velocity of the lower stream is zero.

The equation for the KH instability becomes:

$$\frac{\dfrac{{v_{\mathrm{ph}}'}^2}{c_1^2} - 1}{\dfrac{\gamma_1^2 {v_{\mathrm{ph}}'}^4}{c_1^4}} = \frac{\dfrac{(v_{\mathrm{ph}}' - \Delta v \cos\phi)^2}{c_2^2} - 1}{\dfrac{\gamma_2^2 (v_{\mathrm{ph}}' - \Delta v \cos\phi)^4}{c_2^4}} \tag{75}$$

Finally, we express all velocities in terms of ratios with respect to the sound speed in medium 1, i.e.

$$x = \frac{v_{\mathrm{ph}}'}{c_1} \qquad m = \frac{\Delta v \cos\phi}{c_1} \tag{76}$$

and this gives

$$\frac{x^2 - 1}{x^4} = \frac{\gamma_1^2}{\gamma_2^2}\frac{c_2^2}{c_1^2}\left[\frac{(x-m)^2 - c_2^2/c_1^2}{(x-m)^4}\right] \qquad (77)$$

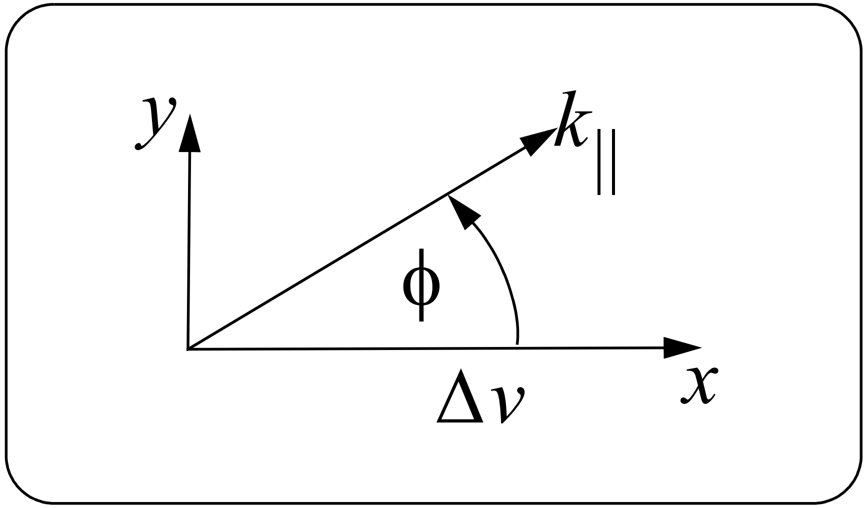
This is the basic dispersion relation for the compressible KH instability. It is a sixth order polynomial equation when multiplied out.

Condition on the solution

Since we want the solutions to vanish at $z = \pm\infty$ then we have the following important conditions on the solution:

$$Re(k_{1,z}) < 0 \qquad \text{and} \qquad Re(k_{2,z}) > 0 \qquad (78)$$

3.3 Physical interpretation of x and m



$$x = \frac{v_{ph}'}{c_1} = \text{Phase velocity of perturbation in frame of lower stream relative to speed of sound}$$

$$m = \frac{\Delta v \cos\phi}{c_1} = \text{Relative Mach number of 2 streams in direction of perturbation} \tag{79}$$

3.4 Special cases

3.4.1 Polytropic indx identical in both streams

$$\gamma_1 = \gamma_2 \tag{80}$$

Take

$$a = \frac{c_2}{c_1} = \text{Ratio of sound speeds} \tag{81}$$

then the dispersion equation becomes

$$\frac{x^2 - 1}{x^4} = a^2 \frac{(x-m)^2 - a^2}{(x-m)^4}$$

$$\frac{1}{x^2} - \frac{1}{x^4} - \frac{a^2}{(x-m)^2} + \frac{a^4}{(x-m)^4} = 0 \qquad (82)$$

$$\left[\frac{1}{x^2} - \frac{a^2}{(x-m)^2}\right] - \left[\frac{1}{x^4} - \frac{a^4}{(x-m)^4}\right] = 0$$

Factoring the second term on the right hand side:

$$\left[\frac{1}{x^2} - \frac{a^2}{(x-m)^2}\right] - \left[\frac{1}{x^2} + \frac{a^2}{(x-m)^2}\right]\left[\frac{1}{x^2} - \frac{a^2}{(x-m)^2}\right] = 0 \quad (83)$$

and factorising the equation:

$$\left[\frac{1}{x^2} - \frac{a^2}{(x-m)^2}\right]\left[1 - \frac{1}{x^2} - \frac{a^2}{(x-m)^2}\right] = 0 \qquad (84)$$

so that either

$$\frac{1}{x^2} = \frac{a^2}{(x-m)^2} \qquad \text{(quadratic)} \qquad (85)$$

or

$$\frac{1}{x^2} + \frac{a^2}{(x-m)^2} = 1 \qquad \text{(quartic)} \qquad (86)$$

Roots of quadratic

$$
\begin{aligned}
a^2x^2 &= (x-m)^2 = x^2 - 2mx + m^2 \\
\Rightarrow (a^2-1)x^2 + 2mx - m^2 &= 0 \qquad (87) \\
\Rightarrow x &= \frac{m}{1-a}, \frac{m}{1+a}
\end{aligned}
$$

Since

$$
m = \frac{\Delta v \cos\phi}{c_1} \qquad a = \frac{c_2}{c_1} \qquad (88)
$$

then

$$
\begin{aligned}
x &= \frac{\omega}{k_{\|} c_1} = \frac{\Delta v \cos\phi}{c_2 - c_1}, \frac{\Delta v \cos\phi}{c_1 + c_2} \\
\frac{\omega}{k_{\|}} &= \frac{c_1}{c_1 - c_2} \Delta v \cos\phi, \frac{c_1}{c_1 + c_2} \Delta v \cos\phi
\end{aligned}
\tag{89}
$$

Both of these roots are real and therefore neither correspond to an instability.

Roots of quartic

$$
\frac{1}{x^2} + \frac{a^2}{(x-m)^2} = 1 \tag{90}
$$

Special case: Equal densities => equal sound speeds

$$a = \frac{c_2}{c_1} = 1$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{(x-m)^2} = 1 \tag{91}$$

$$\Rightarrow (x-m)^2 + x^2 = x^2(x-m)^2$$

In order to solve this equation, put

$$y = x - \frac{m}{2}$$

$$\Rightarrow \left(y - \frac{m}{2}\right)^2 + \left(y + \frac{m}{2}\right)^2 = \left(y + \frac{m}{2}\right)^2\left(y - \frac{m}{2}\right)^2 \qquad (92)$$

$$\Rightarrow 2y^2 + \frac{m^2}{2} = \left(y^2 - \frac{m^2}{4}\right)^2$$

making the equation into the following quadratic in y^2:

$$2y^2 + \frac{m^2}{2} = y^4 - \frac{m^2 y^2}{2} + \frac{m^4}{16}$$

$$\Rightarrow y^4 - \left(\frac{m^2}{2} + 2\right) y^2 + \left(\frac{m^4}{16} - \frac{m^2}{2}\right) = 0 \qquad (93)$$

$$\Rightarrow y^2 = \left(1 + \frac{m^2}{4}\right) \pm (1 + m^2)^{1/2}$$

Now the roots for y^2 are always positive when

$$\left(1+\frac{m^2}{4}\right)^2 > 1+m^2$$

$$1+\frac{m^2}{2}+\frac{m^4}{16} > 1+m^2 \qquad (94)$$

$$\Rightarrow m^2 > 8$$

If $y^2 > 0$ then y is real and so is x, so that there is no instability. Hence, the condition for there to be no instability is:

$$m = \frac{\Delta v \cos\phi}{c_1} > \sqrt{8} \qquad (95)$$

On the other hand, the perturbation is unstable, if

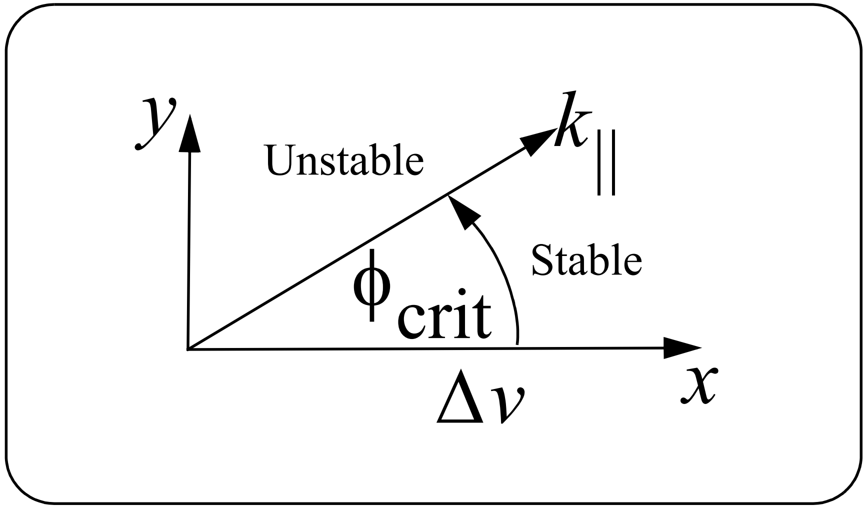
$$M_{\mathrm{rel}} \cos\phi < \sqrt{8} \tag{96}$$

where $M_{\mathrm{rel}} = \dfrac{\Delta v}{c_1}$ is the relative Mach number of the two streams.

Note that for *any* Mach number there is a critical angle for the wave vector for which instability occurs given by:

$$\cos\phi_{\mathrm{crit}} = \frac{\sqrt{8}}{M_{\mathrm{rel}}} \tag{97}$$

Perturbations with $\phi > \phi_{\mathrm{crit}}$ are unstable.



As the Mach number increases, the unstable waves become closer to being perpendicular to the direction of the relative velocity.