Solutions Astrophysical Gas Dynamics Assignment 6

37. *Jet-driven bubble.*

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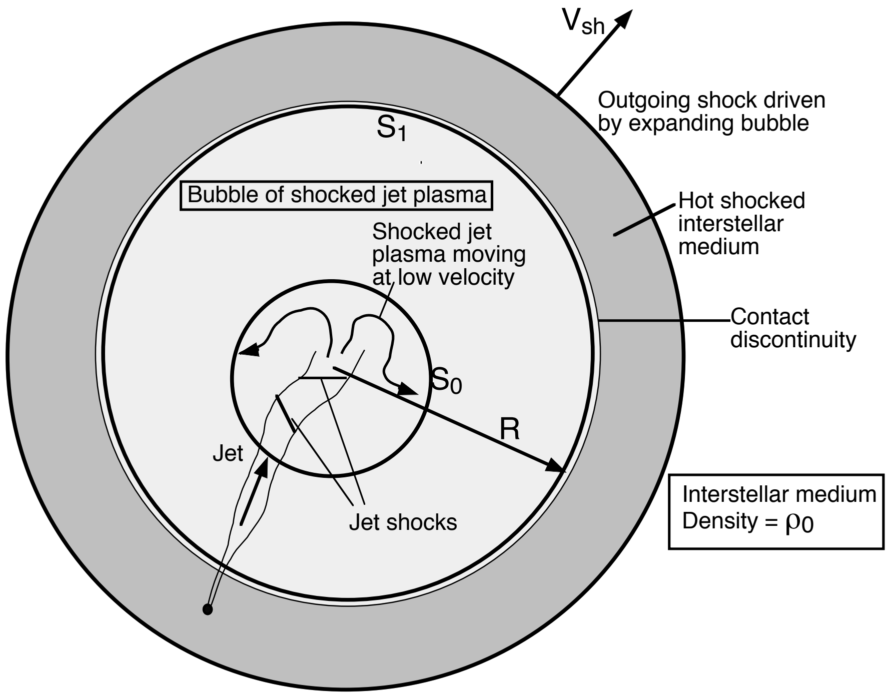

Fig. 1.— Schematic of a jet-inflated bubble.

(a) The equation for the total energy within a moving surface is

$$\frac{\partial}{\partial t}\int_V \left(\epsilon + \frac{1}{2}\rho v^2\right)\, d^3x + \int_S \rho\left(h + \frac{1}{2}v^2\right) v_i n_i\, dS - \int_S \left(\epsilon + \frac{1}{2}\rho v^2\right) u_i n_i\, dS = 0$$

We apply this equation to the volume enclosed by the surfaces S_0 and S_1. The surface S_0 is stationary so that $u_i = 0$ on that surface. The surface S_1 moves outward at the velocity dR/dt The velocity of gas at the surface is also dR/dt. Hence the above equation becomes (taking account of the direction of the normal on S_0):

$$\frac{\partial}{\partial t}\int_V \left(\epsilon + \frac{1}{2}\rho v^2\right) d^3x - \int_{S_0} \rho\left(h + \frac{1}{2}v^2\right) v_i n_i dS + \int_{S_1} \left[\left(\rho h + \frac{1}{2}\rho v^2\right) - \left(\epsilon + \frac{1}{2}\rho v^2\right)\right] u_i n_i dS = 0$$

Since $\rho h - \epsilon = p$ and the integral over S_0 is the energy flux through S_0 then

$$\frac{\partial}{\partial t}\int_V \left(\epsilon + \frac{1}{2}\rho v^2\right)\, d^3x + \int_{S_1} p u_i n_i\, dS = F_E$$

The energy flux of the jet through the surface S_0 is given by

$$F_E = \int_{S_0} \rho\left(h + \frac{1}{2}v^2\right) v_i n_i\, dS$$

assuming non-relativistic flow when the disrupted jet material flows through S_0.

(b) We assume that the total energy density within the bubble is dominated by the internal energy density $\epsilon = 3p$ and that S_1 is approximately spherical, with radius R. Then, the energy within the bubble

$$\int_V \left(\epsilon + \frac{1}{2}\rho v^2\right) d^3x \approx 3p \times \frac{4\pi}{3}R^3 = 4\pi p R^3$$

and

$$\int_{S_1} p u_i n_i \, dS \approx p \times \frac{dR}{dt} \times 4\pi R^2 = 4\pi p R^2 \frac{dR}{dt}$$

so that

$$\begin{aligned}
\frac{d}{dt}\left[4\pi p R^3\right] + 4\pi p R^2 \frac{dR}{dt} &= F_E \\
\Rightarrow \frac{d}{dt}(pR^3) + pR^2 \frac{dR}{dt} &= \frac{F_E}{4\pi} \\
\Rightarrow R^3 \frac{dp}{dt} + 4pR^2 \frac{dR}{dt} &= \frac{F_E}{4\pi}
\end{aligned}$$

(c) We have a shock outside the bubble, propagating into the external medium with velocity $v_{\rm sh}$. Assuming that the shock is strong, the post-shock velocity, $v_{\rm ps}$, is:

$$\begin{aligned}
v_{\rm ps} &= \frac{3}{4} v_{\rm sh} \\
\Rightarrow \frac{dR}{dt} &= \frac{3}{4} v_{\rm sh} \\
\Rightarrow v_{\rm sh} &= \frac{4}{3}\frac{dR}{dt}
\end{aligned}$$

The pressure of the shocked, $\gamma = 5/3$ external medium is

$$p_{\rm sh} = \frac{3}{4}\rho_{\rm ext} v_{\rm sh}^2 = \frac{4}{3}\rho_{\rm ext}\left(\frac{dR}{dt}\right)^2$$

(d) The interface between the shocked interstellar gas and the bubble of relativistic gas is a contact discontinuity, so that the pressure is continuous across this interface. Hence the equations describing the evolution of the bubble are:

$$\begin{aligned}
\Rightarrow R^3 \frac{dp}{dt} + 4pR^2 \frac{dR}{dt} &= \frac{F_E}{4\pi} \\
p &= \frac{4}{3}\rho_{\rm ext}\left(\frac{dR}{dt}\right)^2
\end{aligned}$$

We assume power-law expressions for the radius and pressure:

$$\begin{aligned}
R &= a_1 t^{\alpha_1} \Rightarrow \frac{dR}{dt} = \alpha_1 a_1 t^{\alpha_1 - 1} \\
p &= a_2 t^{\alpha_2} \Rightarrow \frac{dpR^3}{dt} = (3\alpha_1 + \alpha_2) a_1 \, a_2^3 \, t^{3\alpha_1 + \alpha_2 - 1}
\end{aligned}$$

Substituting in to the two equations for p and R:

$$\begin{aligned}
(3\alpha_1 + \alpha_2) a_1 \, a_2^3 \, t^{3\alpha_1 + \alpha_2 - 1} + \alpha_1 \, a_1^3 \, a_2 \; t^{3\alpha_1 + \alpha_2 - 1} &= F_E \\
a_2 \, t^{\alpha_2} &= \frac{4}{3}\rho_{\rm ext} a_1^2 \, \alpha_1^2 t^{2\alpha_1 - 2}
\end{aligned}$$

Equating powers of t:

$$\begin{aligned} 3\,\alpha_1 + \alpha_2 - 1 &= 0 \\ \alpha_2 &= 2\alpha_1 - 2 \end{aligned}$$

Solution of these equations gives

$$\alpha_1 = \frac{3}{5} \qquad \alpha_2 = -\frac{4}{5}$$

The coefficients of the powers of t give:

$$\begin{aligned} (3\alpha_1 + \alpha_2)\,(a_1^3\,a_2) + \alpha_1\,(a_1^3\,a_2) = (4\alpha_1 + \alpha_2)\,(a_1^3\,a_2) &= \frac{F_E}{4\pi} \\ a_2 &= \frac{4}{3}\rho_{\rm ext}\,\alpha_1^2\,a_1^2 \end{aligned}$$

The solution to these equations is:

$$\begin{aligned} a_1 &= \left[\frac{5^3}{3\times 2^7\pi}\,\frac{F_E}{\rho_{\rm ext}}\right]^{1/5} \\ a_2 &- \frac{12}{25}\,\rho_{\rm ext}\,a_1^2 = \frac{12}{25}\,\rho_{\rm ext}\left[\frac{5^3}{3\times 2^7\pi}\,\frac{F_E}{\rho_{\rm ext}}\right]^{2/5} \end{aligned}$$

(e) With $F_E = 10^{37}$ W, $n = 10^4\ {\rm m}^{-3}$ and $t = 10^6$ yr, $R \approx 2.6$ kpc

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38. *Magnetic field in a neutron star.*

(a) Suppose that the geometrical configuration of the magnetic field does not change significantly during the course of the collapse of the star and consider the flux of magnetic field through an element of its surface. Let $\mathbf{B}_0$ be the magnetic flux density at time $t = 0$ and let the area of the element of surface be dS_0. The flux through this element is $B_{n0}dS_0$. When the star has collapsed, use subscripts 1 to refer to the same quantities. The flux though the elementary surface which has contracted to dS_1 is now $B_{n1}dS_1$. The areas of the elements of surface scale with the Radius^2 os the star and since the flux through these elements is conserved, then

$$B_{n0}\,dS_0 = B_{n1}\,dS_1 \Rightarrow B_{n1} = B_{n0}\,\frac{dS_0}{dS_1} = B_{n0}\,\frac{R_0^2}{R_1^2}$$

where R is the radius. Since we are assuming that the magnetic field retains the same geometrical configuration, then $B_n \propto B$ and the stellar magnetic field is given by:

$$B_1 = B_0\,\frac{R_0^2}{R_1^2}$$

at each point of the star's surface.

(b) For the numerical values in the question:

$$B_{NS} = 1 \times \left(\frac{10\times 6.96\times 10^{10}}{10^5}\right)^2\ \text{Gauss} = 4.8\times 10^{11}\ \text{Gauss}$$

39. *Magnetic field in a jet.*

Consider an elementary section of jet defined by the (r, ϕ, z) coordinate lines as shown in figure 2. The fluxes through the r, ϕ and z coordinate faces are given by

$$\begin{aligned} \Phi_r &= B_r\, r\delta\phi\, \delta z \\ \Phi_\phi &= B_\phi\, \delta r\, \delta z \\ \Phi_z &= B_z\, r\delta\phi\delta r \end{aligned}$$

These fluxes are conserved as the gas is advected along the jet. We therefore have to determine how the various sides of the elementary volume change. We assume that the expansion of the jet is homologous so that the radius of the element $r \propto R(z)$ where $R(z)$ is the jet radius at z. The ϕ coordinate of the edges of the section do not change so that $\delta\phi$ does not change as the element moves along the jet.

(a) Consider the equation for streamline separation:

$$\frac{d(\delta x_i)}{dt} = v_{i,j}\delta x_j$$

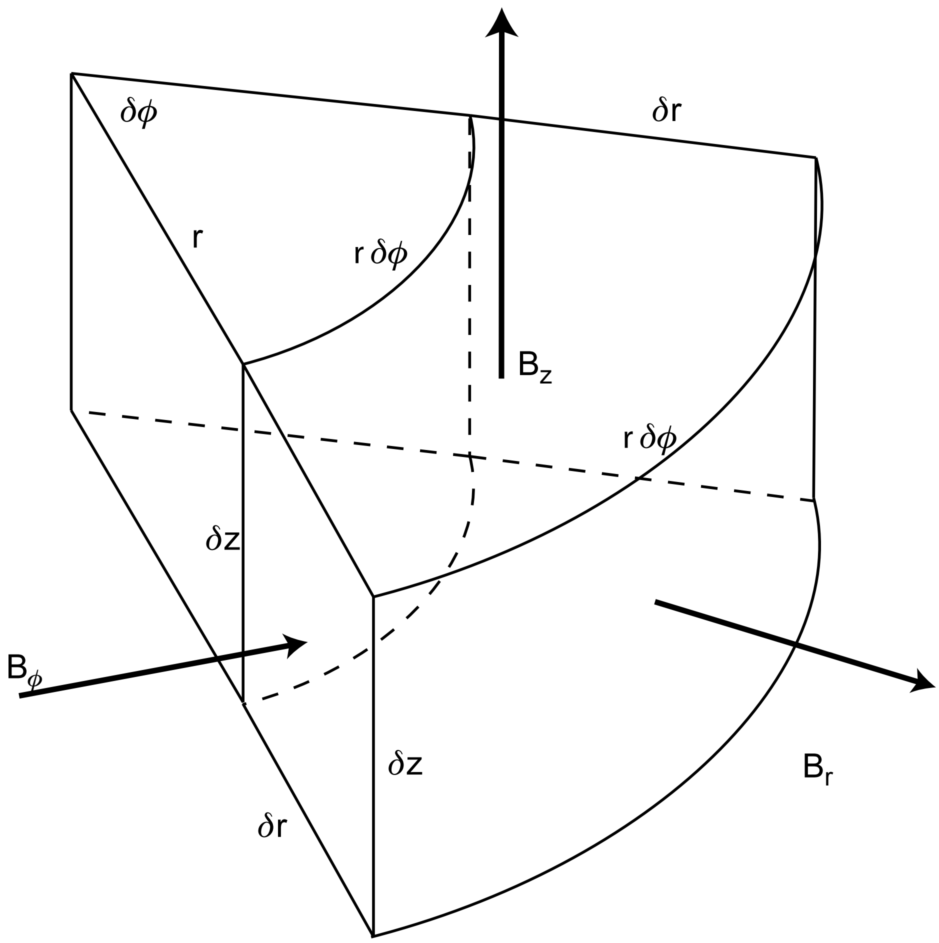

Fig. 2.— Elementary section of a jet.

which is, in dyadic form:

$$\frac{\partial}{\partial t}\delta\mathbf{x} + \mathbf{v}\cdot\nabla\delta\mathbf{x} = \delta\mathbf{x}\cdot\nabla\mathbf{v}$$

If we consider a streamline displacement with r, ϕ and z components, then

$$\delta\mathbf{x} = \delta r\,\hat{\mathbf{r}} + (r\delta\phi)\,\hat{\phi} + \delta z\,\hat{\mathbf{z}}$$

The full expression for $\mathbf{A}\cdot\nabla\mathbf{B}$ is give in the Wikipedia page:
https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates.
Taking $\mathbf{v} = v_r\,\hat{\mathbf{r}} + v_z\hat{\mathbf{z}}$, (i.e. $v_\phi = 0$; non-rotating jet) we obtain, in cylindrical coordinates:

$$\begin{aligned}
\frac{\partial}{\partial t}\delta r + v_r\,\frac{\partial\delta r}{\partial r} + v_z\,\frac{\partial\delta r}{\partial z} &= \delta r\,\frac{\partial v_r}{\partial r} \\
\frac{\partial(r\delta\phi)}{\partial t} + v_r\,\frac{\partial(r\delta\phi)}{\partial r} + v_z\,\frac{\partial(r\delta\phi)}{\partial z} &= v_r\,\delta\phi \\
\frac{\partial\delta z}{\partial t} + v_r\,\frac{\partial\delta z}{\partial r} + v_z\,\frac{\partial\delta z}{\partial z} &= \delta z\,\frac{\partial v_z}{\partial z}
\end{aligned}$$

We first consider the implications of these equations for the relative separation of two points separated by $\delta\mathbf{x} = (0,0,\delta z)$. The first and second separation equations show that for $\delta r = \delta\phi = 0$, initially, δr and $\delta\phi$ remain zero.
For a steady jet, $(\partial/\partial t = 0)$, and neglecting v_r in comparison with v_z, then

$$\begin{aligned}
v_z\frac{\partial\delta z}{\partial z} &= \delta z\,\frac{\partial v_z}{\partial z} \\
\Rightarrow \frac{1}{\delta z}\frac{\partial\delta z}{\partial z} &= \frac{1}{v_z}\frac{\partial v_z}{\partial z} \\
\Rightarrow \delta z &= \text{Constant}\times v_z
\end{aligned}$$

This implies that when the jet accelerates, the separation between points increases, and when it decelerates, the separation decreases.
We now consider the evolution of points separated in radial coordinate by δr. We are assuming that the jet expands homologously so that for a point initially at $r = r_0, z = z_0$ its radial coordinate at z is given by:

$$\frac{r}{R(z)} = \frac{r_0}{R(z_0)} \Rightarrow r = \frac{r_0}{R(z_0)}\,R(z)$$

and the evolution of the separation between points which are initially at the same height and separated by δr_0 is given by

$$\delta r = \delta r_0\frac{R(z)}{R(z_0)}$$

i.e the separation in creases by the factor $R(z)/R(z_0)$. This can also be shown to be consistent with the streamline separation for δr.
Finally, consider the equation for the separation when $\delta r = \delta z = 0$ and $\delta\phi \neq 0$. When the flow is steady, the equation for $\delta\phi$ can be written

$$r v_r\,\frac{\partial(\delta\phi)}{\partial r} + v_z\,r\frac{\partial(\delta\phi)}{\partial z} = r\frac{d\delta\phi}{dt} = 0.$$

Hence the angular separation does not change in this case.

(b) Returning now to the expressions for the fluxes, we have:

$$\begin{aligned}\Phi_r &\propto B_r R(z) v_z(z) \\ \Phi_\phi &\propto B_\phi R(z) v_z(z) \\ \Phi_z &\propto B_z R^2(z)\end{aligned}$$

All of these fluxes are conserved, so that, putting $v_z(z) = V(z)$:

$$\begin{aligned}B_r(z)R(z)V(z) &= B_r(z_0)R(z_0)V(z_0) \quad \Rightarrow \quad B_r(z) = B_r(z_0)\frac{R(z_0)V(z_0)}{R(z)V(z)} \\ B_\phi(z)R(z)V(z) &= B_\phi(z_0)R(z_0)V(z_0) \quad \Rightarrow \quad B_\phi(z) = B_\phi(z_0)\frac{R(z_0)V(z_0)}{R(z)V(z)} \\ B_z(z)R^2(z) &= B_z(z_0)R^2(z_0) \quad \Rightarrow \quad B_z(z) = B_z(z_0)\frac{R^2(z_0)}{R^2(z)}\end{aligned}$$

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41. *Mass, momentum and energy flux.*

We have:

$$\begin{aligned} M &= 2 & v &= 20,000 \text{ km s}^{-1} = 2 \times 10^9 \text{ cm s}^{-1} \\ p &= 10^{-11} \text{ dynes cm}^{-2} & B^2/8\pi &= 10^{-11} \text{ dynes cm}^{-2}\end{aligned}$$

Sound speed: $M = v/c_s = 2 \Rightarrow c_s = v/M = 10^9 \text{ cm s}^{-1}$.

(a) Jet temperature:

$$c_s = \sqrt{\frac{\gamma k T}{\mu m}} \Rightarrow T = \frac{\mu m c_s^2}{\gamma k} = 4.47 \times 10^9 \ K$$

(b) Number density:

$$p = nkT \Rightarrow n = \frac{p}{kT} = 1.62 \times 10^{-5} \text{ cm}^{-3}$$

Also, for future reference:

$$\rho = \mu m n = 1.67 \times 10^{-29} \text{ gm cm}^{-3}$$

(c) Mass flux:

$$\dot{M} = \rho v \times A_{\rm jet}$$

where $A_{\rm jet}$ = Jet cross-sectional area $= \pi R_{\rm jet}^2 \approx 3.00 \times 10^{39} \text{ cm}^2$ for $R_{\rm jet} = 10$ pc. Hence

$$\dot{M} = 1.6 \times 10^{-6} \ M_\odot \text{ yr}^{-1}$$

(d) Momentum flux:

$$\begin{aligned}\text{Momentum flux density} &= (\rho v_i v_j + p\delta_{ij} - M_{ij})\, n_j \\ &= \rho v_i v_j n_j + p n_i - \frac{B_i B_j n_j}{4\pi} + \frac{B^2}{8\pi} n_i\end{aligned}$$

Let n_j and v_j be in the z-direction: $n_i = (0, 0, 1)$, then $B_z = 0$ since the magnetic field is perpendicular to to the jet, and

$$\text{Momentum flux} = (\rho v^2 + p + \frac{B^2}{8\pi}) \times A_{\text{jet}}$$

Inserting the above numerical values gives Momentum flux density $= 8.68 \times 10^{-11}$dynescm^{-2} and the momentum flux (the force exerted by the jet) is 2.6×10^{29} dynes.

(e) Energy flux:

$$F_E = \left[\left(\frac{1}{2} \rho v^2 + \rho h \right) v + \frac{B^2}{4\pi} v_\perp \right] \times A_{\text{jet}}$$

Since the magnetic field is perpendicular to the velocity, $v_\perp = v$. Inserting numerical values gives

$$F_E = 4.7 \times 10^{38} \text{ ergs s}^{-1}$$