Solutions to Astrophysical Gas Dynamics Assignment 5

23. *Bernoulli's equation.*

(i) Start with the momentum equations in the form:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_j}{\partial x_j} = \frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i}$$

Divide by the density and take the scalar product with v_i:

$$v_i \frac{\partial v_i}{\partial t} + v_j v_i \frac{\partial v_j}{\partial x_j} = \frac{1}{\rho} v_i \frac{\partial p}{\partial x_i} - v_i \frac{\partial \phi}{\partial x_i}$$

Now take the scalar product of this equation with v_i using:

$$\begin{aligned} v_i \frac{\partial v_i}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) \\ v_i v_j \frac{\partial v_i}{\partial x_j} &= v_j \frac{\partial}{\partial x_j}\left(\frac{v^2}{2}\right) \end{aligned}$$

Also use the entropy equation:

$$\begin{aligned} kT ds &= dh - \frac{dp}{\rho} \\ \Rightarrow kT v_i \frac{\partial s}{\partial x_i} &= v_i \frac{\partial h}{\partial x_i} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \end{aligned}$$

(ii) Substituting in the above equation:

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + v_j \frac{\partial}{\partial x_j}\left(\frac{v^2}{2}\right) = kT v_i \frac{\partial s}{\partial x_i} - v_i \frac{\partial h}{\partial x_i} - v_i \frac{\partial \phi}{\partial x_i}$$

Now if the flow is stationary (time-independent), then $\partial s/\partial t = \partial h/\partial t = \partial \phi/\partial t = 0$ and

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = kT\frac{ds}{dt} - \frac{dh}{dt} + \frac{d\phi}{dt}$$

When the flow is adiabatic, $ds/dt = 0$, and we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{v^2}{2} + h + \phi\right) &= 0 \\ \Rightarrow \frac{v^2}{2} + h + \phi &= \text{Streamline constant} \end{aligned}$$

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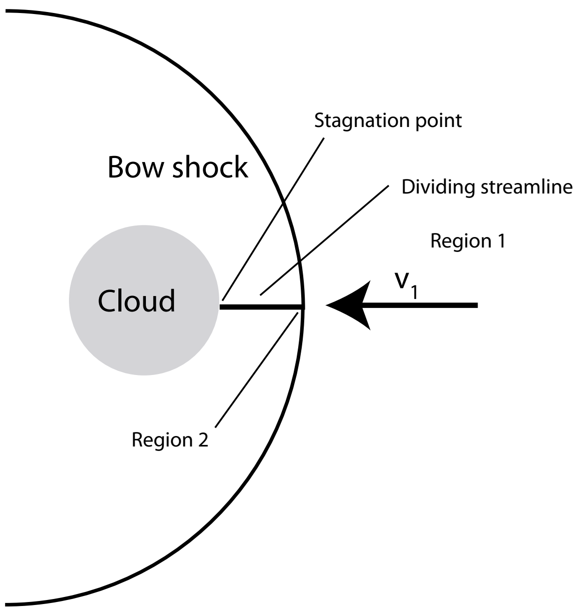
24. *Stagnation pressure of a cloud shock.*

(a) Let ρ_1, p_1, v_1 be the density, pressure and velocity of the cloud in the ambient medium and let ρ_2, p_2, v_2 be the post-shock pressure. We analyse the physics in the frame of the cloud and shock. The stagnation pressure is determined from Bernoulli's equation. Assuming adiabatic post-shock flow:

$$\frac{1}{2}v^2 + h = \text{Streamline Constant} = \frac{1}{2}v_2^2 + h_2$$



where the specific enthalpy

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}.$$

At the stagnation point, $v = 0$ and we obtain for the stagnation specific enthalpy h_s:

$$\begin{aligned} h_s &= \frac{v_2^2}{2} + h_2 \\ \Rightarrow \frac{h_s}{h_2} &= 1 + \frac{1}{2}\frac{v_2^2}{h_2} = 1 + \frac{\gamma - 1}{2}M_2^2 \end{aligned}$$

where M_2 is the post-shock Mach number.

(b) The relationships between the post-shock and pre-shock Mach numbers M_2 and M_1, and the relationship between the post-shock pressure p_2 and pre-shock Mach number are:

$$\begin{aligned} M_2^2 &= \frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)} \\ \frac{p_2}{p_1} &= \frac{2\gamma M_1^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \end{aligned}$$

M_1 is the Mach number of the cloud in the interstellar medium.
Therefore, for a strong shock ($M_1 \to \infty$):

$$M_2^2 = \frac{\gamma - 1}{2\gamma}$$

Substituting in to the above equation for h_s:

$$\frac{h_s}{h_2} = 1 + \frac{\gamma - 1}{2} \times \frac{\gamma - 1}{2\gamma} = \frac{(\gamma + 1)^2}{4\gamma}$$

(c) Since, in the post-shock region,

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{\gamma}{\gamma - 1}K(s)\rho^{\gamma - 1}$$

and $K(s)$ is constant, then

$$\frac{h_s}{h_2} = \left(\frac{\rho_s}{\rho_2}\right)^{\gamma-1} = \left(\frac{p_s}{p_2}\right)^{(\gamma-1)/\gamma}$$

Therefore

$$p_s = p_2\left[1 + \frac{\gamma-1}{2}M_2^2\right]^{\gamma/(\gamma-1)}$$

The post-shock pressure for a strong shock ($M_1 \to \infty$) is $p_2 = 2/(\gamma+1)\rho_1 v_1^2$ and the square bracket terms has been evaluated above. Hence,

$$p_s = \frac{2}{\gamma+1}\rho_1 v_1^2 \left[\frac{(\gamma+1)^2}{4\gamma}\right]^{\gamma/(\gamma-1)}$$

(d) For $\gamma = 5/3$

$$p_s \doteq 0.88\rho_1 v_1^2$$

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32. *Interacting winds in planetary nebulae.*

(a) Following the derivation of the shell mass given in lectures, we integrate throughout a volume whose inner and outer surfaces are just inside and just outside the shell. This gives:

$$\frac{d}{dt}\int_{\rm s}\rho d^3x + \int_{\text{Outer surface}}(\rho_{\rm RG}v_{\rm RG} - \rho_{RG}v_{\rm s})\,dS - \int_{\text{Inner surface}}(\rho_{\rm b}v_{\rm b} - \rho_b v_{\rm s})\,dS$$

where subscripts RG, b and s refer to the red giant wind, the bubble and the shell respectively. Since $v_{\rm b} = v_{\rm sh}$ then the inner surface does not contribute to this integral and the only term that does contribute is the integral over the outer surface:

$$\int_{\text{Outer surface}}(\rho_{\rm RG}v_{\rm RG} - \rho_{RG}v_{\rm s})\,dS = -4\pi R_s^2\rho_{\rm RG}(v_{\rm s} - v_{\rm RG})$$

Furthermore, the density of the red giant wind is given by

$$\rho_{\rm RG} = \frac{\dot{M}_{\rm RG}}{4\pi R_s^2}$$

and therefore

$$\frac{dM_s}{dt} = \frac{\dot{M}_{\rm RG}}{v_{\rm RG}}(v_{\rm s} - v_{\rm RG})$$

When the shell velocity is much greater than the velocity of the red giant wind,

$$\begin{aligned}\frac{dM_s}{dt} &= \frac{\dot{M}}{v_{\rm RG}}v_{\rm s} = \frac{\dot{M}}{v_{\rm RG}}\frac{dR_s}{dt}\\ \Rightarrow M_s &= AR_s\end{aligned}$$

where $A = \dot{M}_{\rm RG}/v_{\rm RG}$

(b) The other equations for momentum and energy are as derived in lectures, viz,

$$\begin{aligned}\frac{d}{dt}\left(M_s\frac{dR_s}{dt}\right) &= 4\pi p_b R_s^2 \\ \frac{d}{dt}\left(p_b R_s^3\right) &= \frac{L_w}{2\pi} - 2p_b R_s^2\frac{dR_s}{dt}\end{aligned}$$

We look for power-law solutions of the form:

$$R_s = a_1 t^{\alpha_1} \qquad p_b = a_2 t^{\alpha_2}$$

Substitution in the differential equations gives

$$\begin{aligned}\alpha_2 &= -2 \\ a_2 &= \alpha_1(2\alpha_1-1)\frac{A}{4\pi} \\ 3\alpha_1+\alpha_2-1 &= 0 \\ (3\alpha_1+\alpha_2+2)a_1^3 a_2 &= \frac{L_w}{2\pi}\end{aligned}$$

giving the solutions:

$$\alpha_1 = 1 \quad \alpha = -2 \quad a_1 = \left(\frac{2L_w}{3A}\right)^{1/3} \quad a_2 = \frac{A}{4\pi}$$

(c) For the typical parameters $\dot{M}_w = 10^{-6} M_\odot\,\mathrm{yr}^{-1}$, $v_w = 2000\,\mathrm{km\,s^{-1}}$, $\dot{M}_{\mathrm{RG}} = 10^{-5} M_\odot\,\mathrm{yr}^{-1}$, $v_{\mathrm{RG}} = 10\,\mathrm{km\,s^{-1}}$, $L_w = 1.3\times 29$ W and $A = 6.3\times 10^{13}\mathrm{kg\,m^{-1}}$. Hence, the radius of the planetary nebula is $0.11t_3$pc where t_3 is the time since the onset of the fast wind in units of a thousand years.