Solutions to Assignment 4

16. *Rankine-Hugoniot relations for a polytropic gas.* The two fundamental equations which can be used to derive all the shock relations are the shock adiabat, relating pressure and specific volume, and the relationship between mass flux, pressure and specific volume. The former is, for a polytropic gas,

$$\frac{\tau_2}{\tau_1} = \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}$$

The latter is:

$$j^2 = \rho_1^2 v_1^2 = \rho_2^2 v_2^2 = \frac{p_2 - p_1}{\tau_1 - \tau_2}$$

The mass-flux relation tells us that the square of the Mach number:

$$M_1^2 \stackrel{\text{def}}{=} \frac{\rho_1 v_1^2}{\gamma p_1} = \frac{\tau_1 j^2}{\gamma p_1} = \frac{1}{\gamma}\frac{p_2 - p_1}{p_1}\frac{\tau_1}{\tau_1 - \tau_2} = \frac{1}{\gamma}\frac{p_2 - p_1}{p_1}\frac{1}{1 - \tau_2/\tau_1}$$

The quantity $1 - \tau_2/\tau_1$ on the right hand side is easily expressed in terms of p_2/p_1 using the shock adiabat and the above equation becomes:

$$M_1^2 = \frac{1}{2\gamma}\left[(\gamma-1) + (\gamma+1)\frac{p_2}{p_1}\right]$$

or, solving for p_2/p_1:

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}$$

which is the second of the required relations.

Again, using the shock adiabat, together with $\rho = 1/\tau$, we have:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma-1) + (\gamma+1)\frac{p_2}{p_1}}{(\gamma+1) + (\gamma-1)\frac{p_2}{p_1}}$$

and using the above equation for p_2/p_1 this becomes:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}$$

The temperature is given by $kT/\mu m = p/\rho$ so that

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{\left[2\gamma M_1^2 - (\gamma-1)\right]\left[(\gamma-1)M_1^2 + 2\right]}{(\gamma+1)^2 M_1^2}$$

The expression for the post-shock Mach number is most easily derived using

$$\rho_1^2 v_1^2 = \rho_2^2 v_2^2 \Rightarrow \frac{\rho_2 v_2^2}{p_2} = \frac{\rho_1 v_1^2}{p_1}\frac{p_1}{p_2}\frac{\rho_1}{\rho_2}$$

Hence,

$$M_2^2 \stackrel{\text{def}}{=} \frac{\rho_2 v_2^2}{\gamma p_2} = \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - (\gamma-1)}$$

17. *Strong shocks.* For a strong shock, p_2/p_1 and $M_1 \to \infty$. Taking the limit of the above equation for the density ratio, gives:

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}$$

The expression for p_2 may be derived using the above equation for p_2/p_1. As $M_1 \to \infty$,

$$\frac{p_2}{p_1} \sim \frac{2\gamma}{\gamma+1} M_1^2 = \frac{2}{\gamma+1} \frac{\rho_1 v_1^2}{p_1}$$

Hence,

$$p_2 \sim \frac{2}{\gamma+1} \rho_1 v_1^2$$

The expression for the temperature follows from $kT/\mu m = p/\rho$ giving

$$\frac{kT_2}{\mu m} = \frac{2(\gamma-1)}{(\gamma+1)^2} v_1^2$$

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18. *Temperatures in strong shocks.* From question 6, the temperature following a $\gamma = 5/3$ shock is

$$T_{\rm shock} = \frac{3}{16} \frac{\mu m}{k} v_1^2$$

For $\mu = 0.62$ and velocities of $200\,{\rm km\,s^{-1}}$ and $400\,{\rm km\,s^{-1}}$ this gives temperatures of $5.4 \times 10^5\,K$ and 2.1×10^6 respectively.