Astrophysical Gas Dynamics

TODAY:

Grid-based hydrodynamics

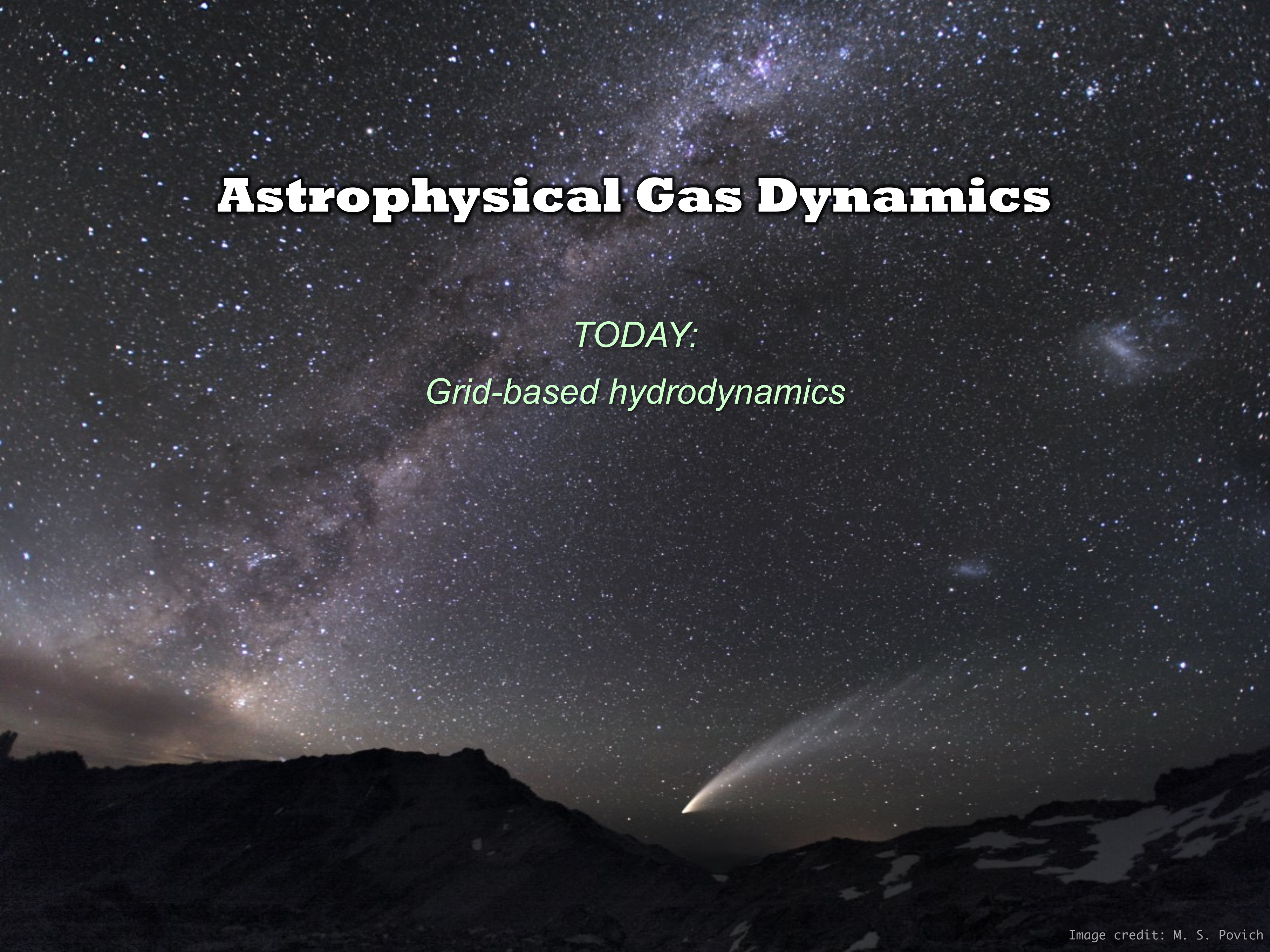
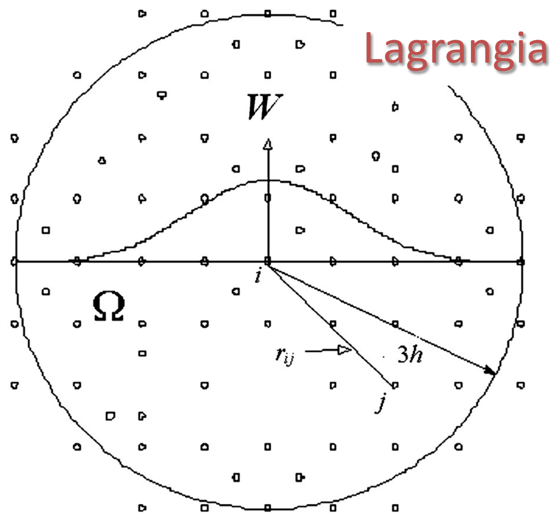


Image credit: M. S. Povich

Fundamentals of SPH and grid

Smoothed Particle Hydrodynamics (SPH)
(Lucy 1977; Gingold & Monaghan 1977)

Lagrangian



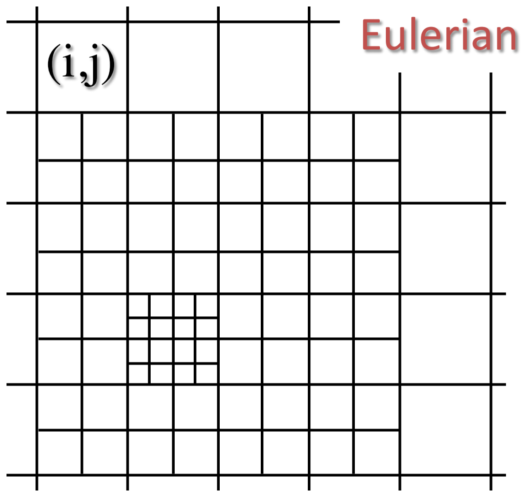
$$\rho(\mathbf{r}) = \sum_b m_b W(\mathbf{r} - \mathbf{r}_b, h)$$

$$\nabla A(\mathbf{r}) = \sum_b m_b \frac{A_b}{\rho_b} \nabla W(\mathbf{r} - \mathbf{r}_b, h)$$

$$W(x,h) = \frac{1}{h\sqrt{\pi}} e^{-(x^2/h^2)}$$

Adaptive Mesh Refinement (AMR)
(Berger & Collela 1989)

Eulerian



- Hydro variables are averages in cells
- Compute fluxes through cell faces
- Simple data structure: indexing
- Finite Volume vs. Finite Difference

R. Leveque: „Nonlinear Conservation Laws and Finite Volume Methods for Astrophysical Fluid Flow“

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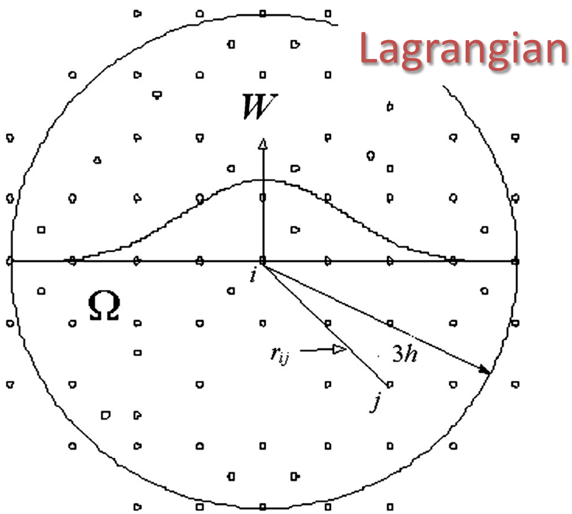


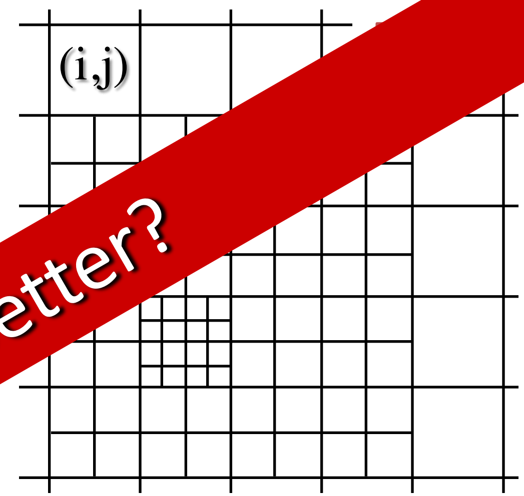
$$\rho(\mathbf{r}) = \sum_b m_b W(\mathbf{r} - \mathbf{r}$$

$$\nabla A(\mathbf{r}) \quad \rho_b \quad \nabla W(\mathbf{r} - \mathbf{r}_b, h)$$

$$W \quad , h) = \frac{1}{h\sqrt{\pi}} e^{-(x^2/h^2)}$$

Which method is better?

Adaptive Mesh Refinement (AMR)
(Berger & Collela 1989)



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- Finite Volume vs. Finite Difference

R. Leveque: „Nonlinear Conservation Laws and Finite Volume Methods for Astrophysical Fluid Flow“

Comparison of SPH and grid in supersonic turbulence

TWO REGIMES OF TURBULENT FRAGMENTATION AND THE STELLAR INITIAL MASS FUNCTION FROM PRIMORDIAL TO PRESENT-DAY STAR FORMATION

Paolo Padoan,[1] Åke Nordlund,[2] Alexei G. Kritsuk,[1] Michael L. Norman,[1] and Pak Shing Li[3]
Received 2006 October 16; accepted 2007 February 16

Their conclusion:

" SPH simulations of large scale star formation to date fail in all three fronts: numerical diffusivity, numerical resolution, and presence of magnetic fields. This should cast serious doubts on the value of comparing predictions based on SPH simulations with observational data (see also Agertz et al. 2006). "

Driven turbulence comparison of SPH and grid

Motivation (role of supersonic turbulence for star formation)

Setup   (Phantom and FLASH):

1. Same initial conditions: uniform density, zero velocities
2. Same turbulence forcing!
3. Driven to Mach number 10
4. Resolutions: 128^3, 256^3 and 512^3 (134,217,728) both grid and SPH

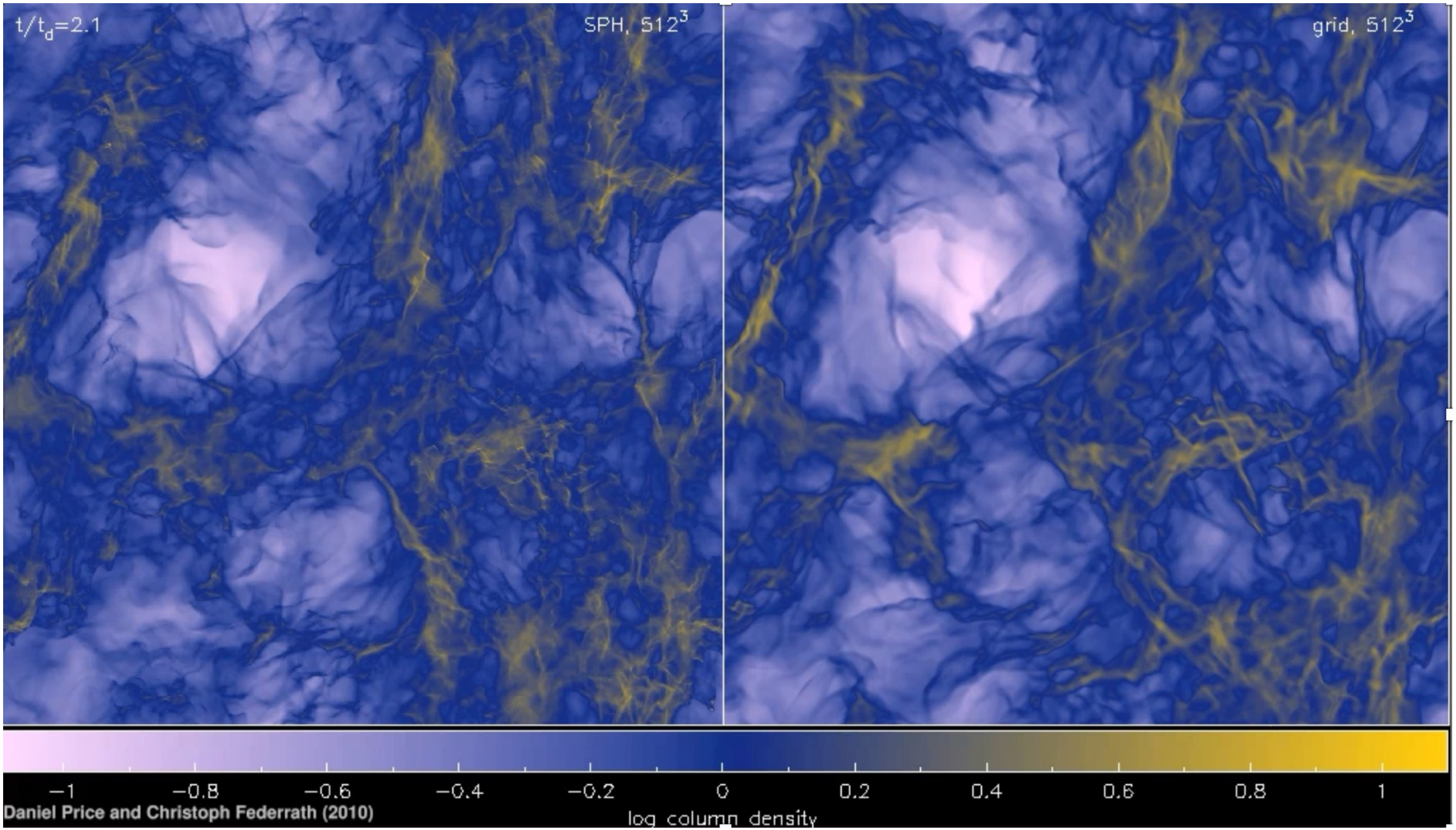
(Price & Federrath 2010, MNRAS 406, 1659)

Driven turbulence comparison of SPH and grid

Phantom

Column density

FLASH

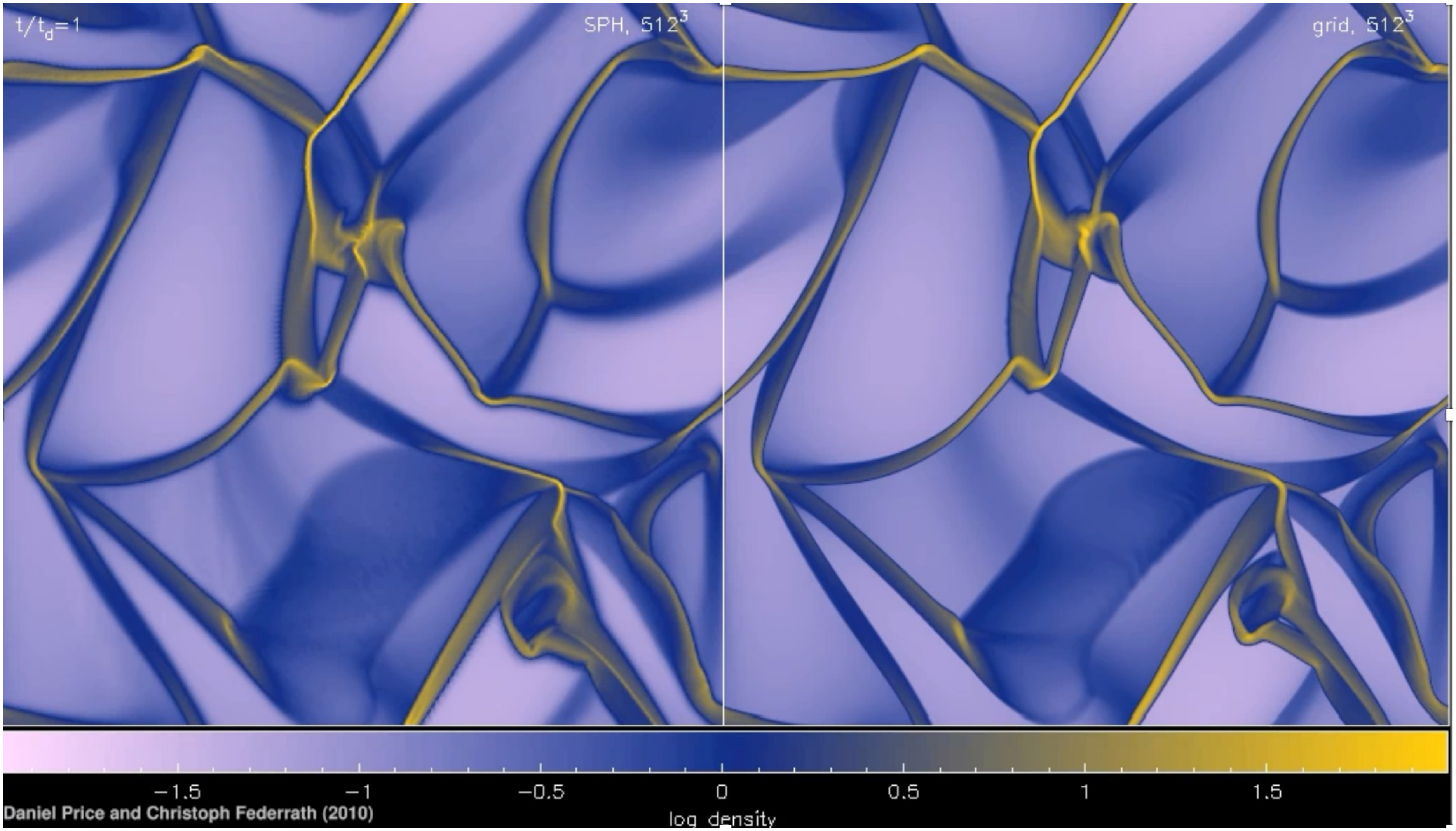


Driven turbulence comparison of SPH and grid

Phantom

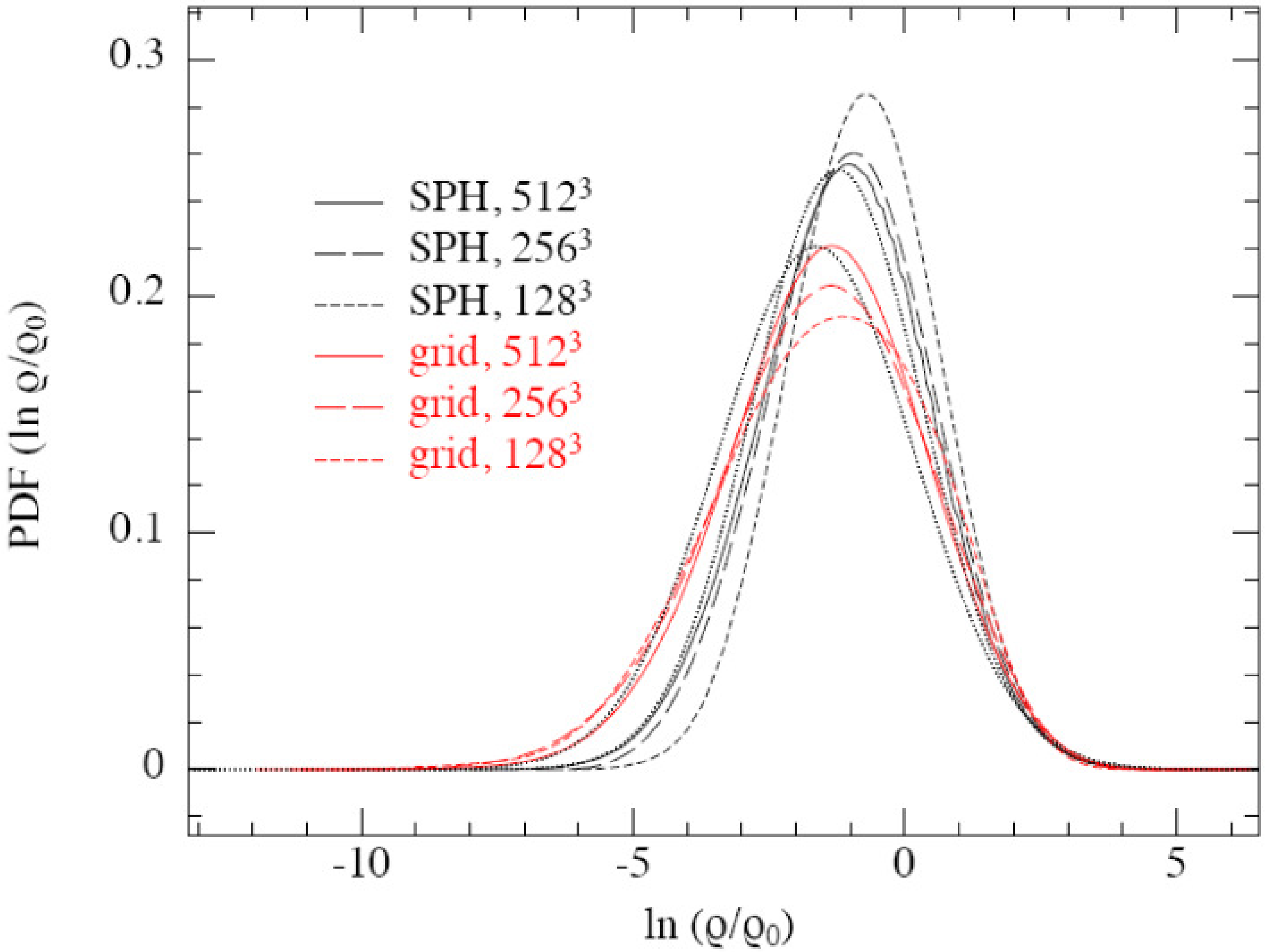
Slices of density

FLASH



Driven turbulence comparison of SPH and grid

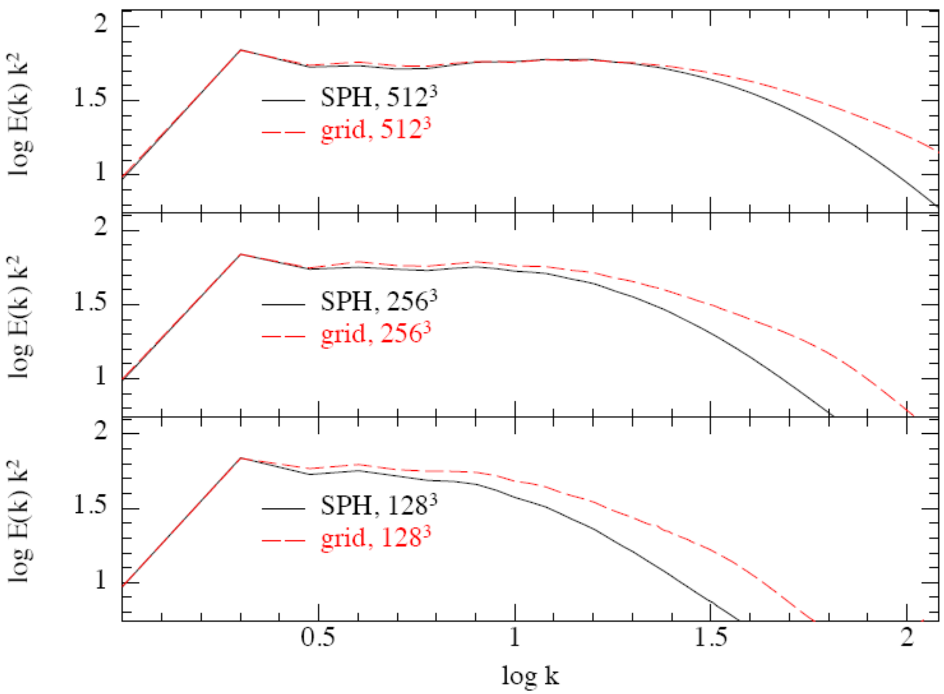
Density Probability Distribution Function (PDF):



PDFs converge with higher resolution

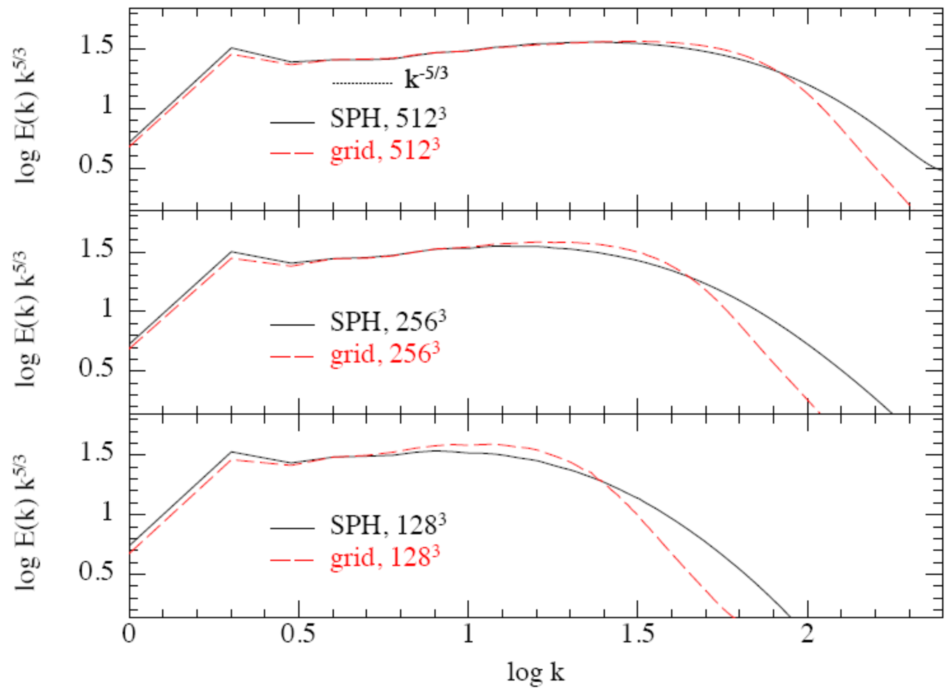
Driven turbulence comparison of SPH and grid

Velocity spectra, v
(VOLUME-weighted)



Grid code less dissipative

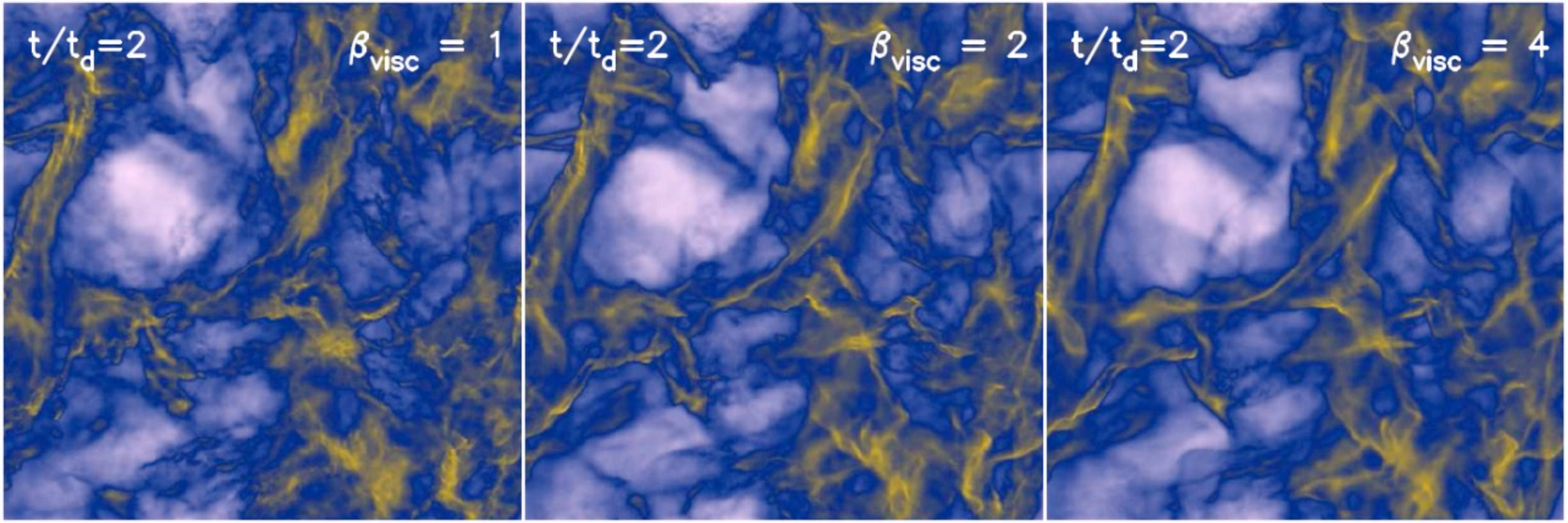
Velocity spectra, $\rho^{1/3}v$
(DENSITY-weighted)



SPH code slightly less dissipative

Driven turbulence comparison of SPH and grid

Influence of β-viscosity in SPH on the modelling of strong shocks



$\beta=1$ $\beta=2$ $\beta=4$

Particle interpenetration for $\beta<4$

Driven turbulence comparison of SPH and grid

Conclusion

(Price & Federrath 2010, MNRAS 406, 1659)

Convergence of SPH and grid

Computational time pure hydro (no gravity):

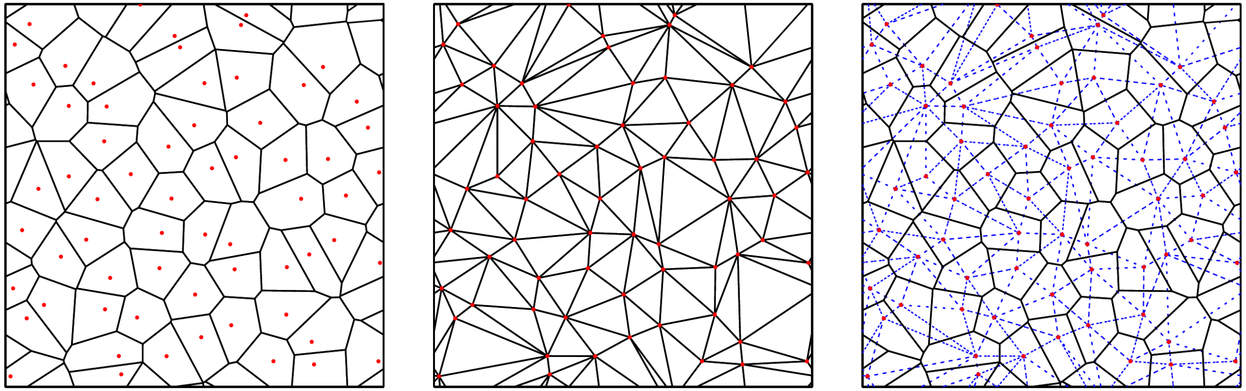
FLASH grid about 20 times faster than Phantom SPH

Strength and Weaknesses of SPH and grid

SPH	Grid (AMR)
+ Automatic refinement on density	+ Simpler data structure (indexing)
+ Typically faster in collapse calculations	+ Typically faster for pure hydro
+ More robust	+ Refinement on arbitrary quantities (e.g., position, shocks, etc.)
+ Intrinsic mass conservation	+ Magnetic fields, shocks, instabilities
- More complex data structure	- Needs more resolution elements for collapse calculations (AMR)
- Potential problems with magnetic fields and/or shocks (see artificial viscosity)	- Sometimes less robust (solver crashes)

Unstructured Grid (e.g. AREPO)

Springel 2010



The basics of grid-based hydrodynamics

1. Advection (Basics, Time stepping, Diffusion, …)
2. Flux conservation and flux limiters
3. Conservative grid-based hydrodynamics
4. Basics of Riemann problem -> Riemann solvers

Lecture based on a lecture given by Kees Dullemond, 2009/2010, Heidelberg

Literature: Randall J. LeVeque, "Finite Volume Methods for Hyperbolic Problems" (Cambridge Texts in Applied Mathematics)

Advection test: code

The basics of grid-based hydrodynamics

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Flux-conserving grid-based hydrodynamics

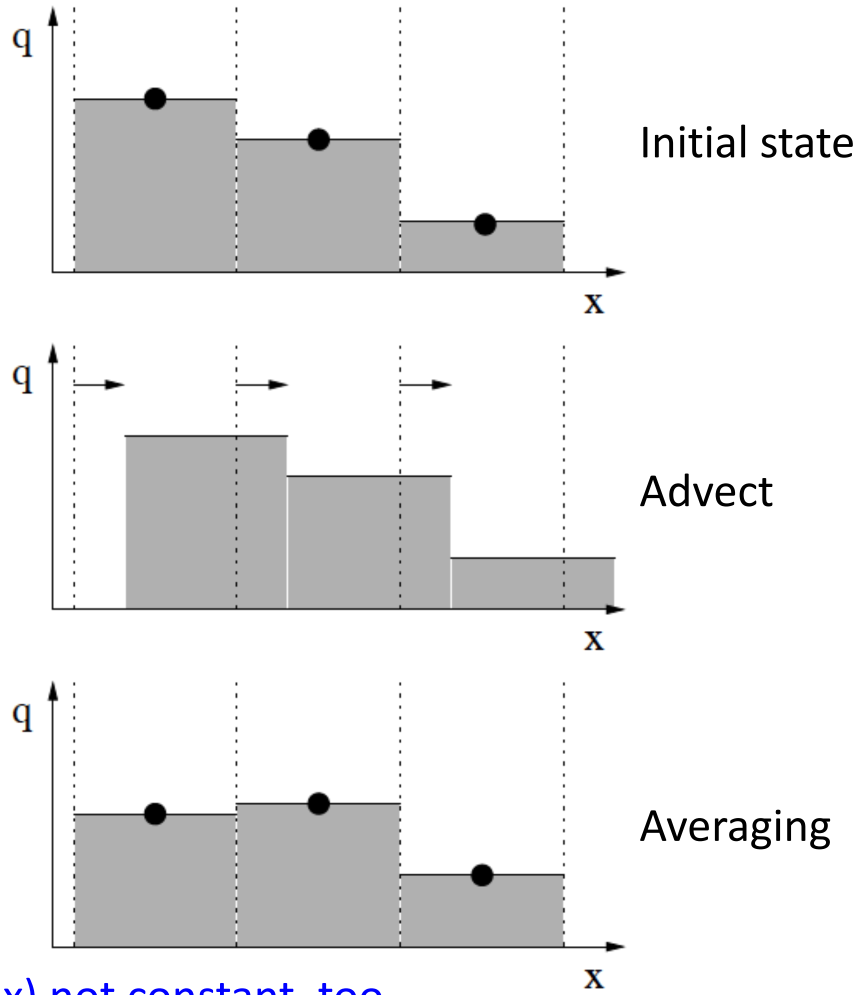
Donor-cell advection:

Piecewise constant subgrid model:

$$\tilde{q}^{n+1/2}_{i+1/2} = \begin{cases} q^n_i & \text{for } u_{i+1/2} > 0 \\ q^n_{i+1} & \text{for } u_{i+1/2} < 0 \end{cases}$$

Flux:

$$f^{n+1/2}_{i+1/2} = \begin{cases} u_{i+1/2}\, q^n_i & \text{for } u_{i+1/2} > 0 \\ u_{i+1/2}\, q^n_{i+1} & \text{for } u_{i+1/2} < 0 \end{cases}$$



Like upwind scheme, but works for u(x) not constant, too.

Flux-conserving grid-based hydrodynamics

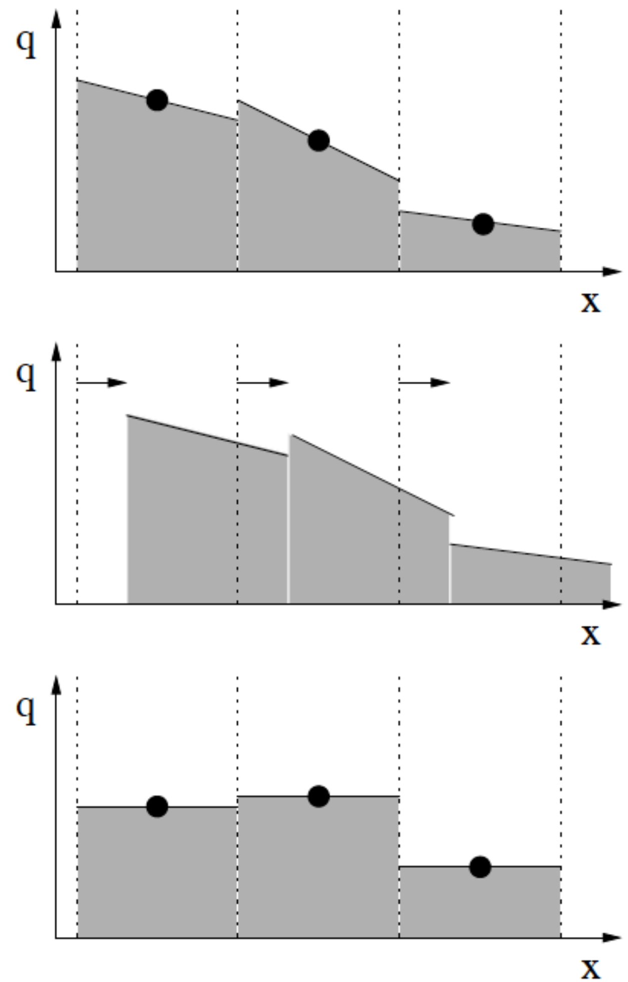
Piecewise linear subgrid model for flux:

- Donor-cell is quite diffusive ->
  Use higher-order subgrid model

$$q(x, t = t_n) = q_i^n + \sigma_i^n (x - x_i)$$

(slope)

Choice of slope



„MUSCL (Monotonic Upwind-centered Scheme for Consveration Laws)“

Flux-conserving grid-based hydrodynamics

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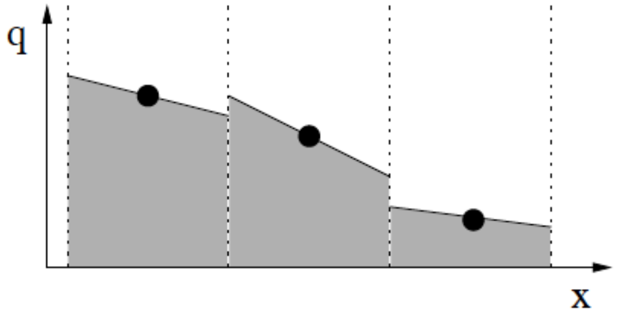


$$q(x, t = t_n) = q_i^n + \sigma_i^n (x - x_i)$$

Different slope choices:

Centered slope: $\sigma_i^n = \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$ (Fromm's method)

Upwind slope: $\sigma_i^n = \frac{q_i^n - q_{i-1}^n}{\Delta x}$ (Beam-Warming method)

Downwind slope: $\sigma_i^n = \frac{q_{i+1}^n - q_i^n}{\Delta x}$ (Lax-Wendroff method)

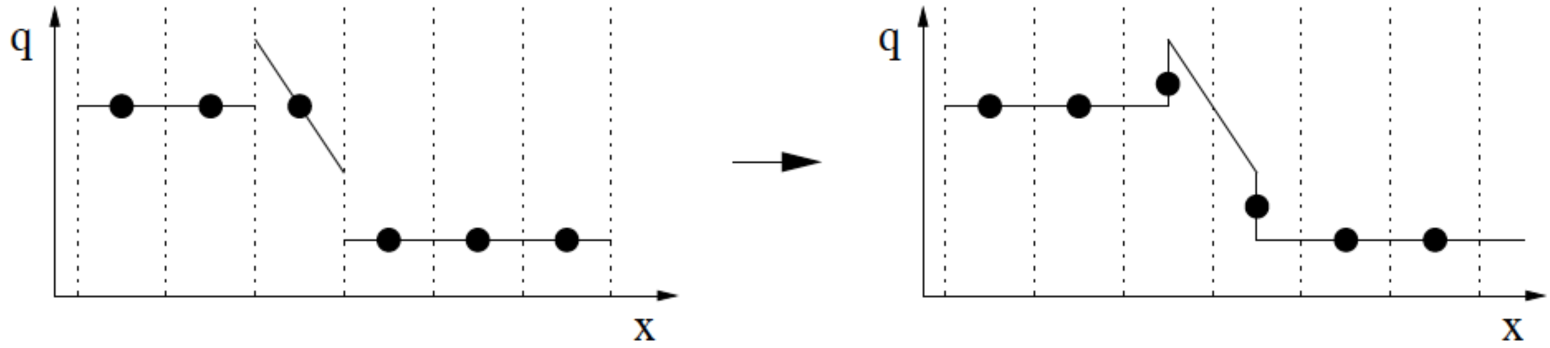
Higher-order now, but beware oscillations

„MUSCL (Monotonic Upwind-centered Scheme for Consveration Laws)“

Flux-conserving grid-based hydrodynamics

Piecewise linear subgrid model for flux:

- can produce overshoots



Piecewise linear subgrid model for flux:

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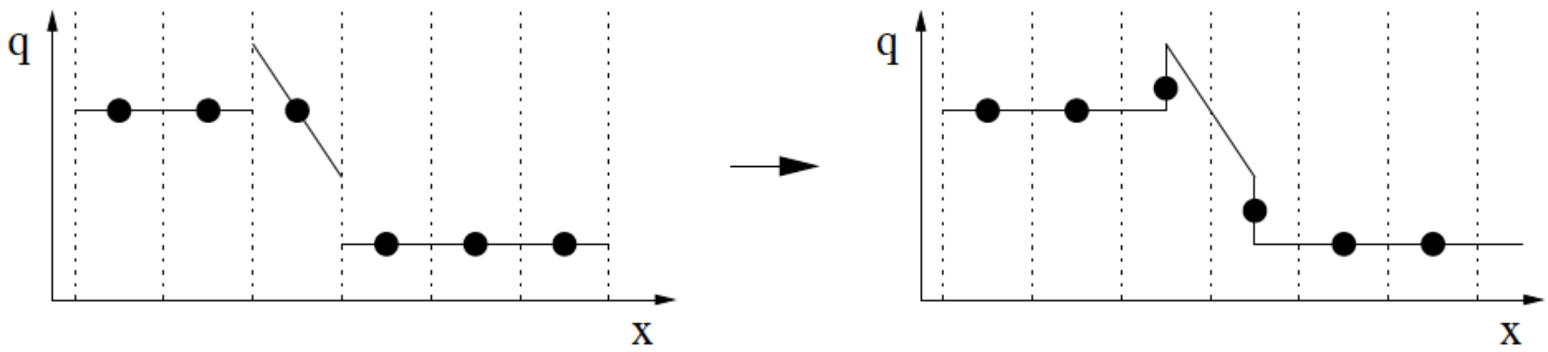
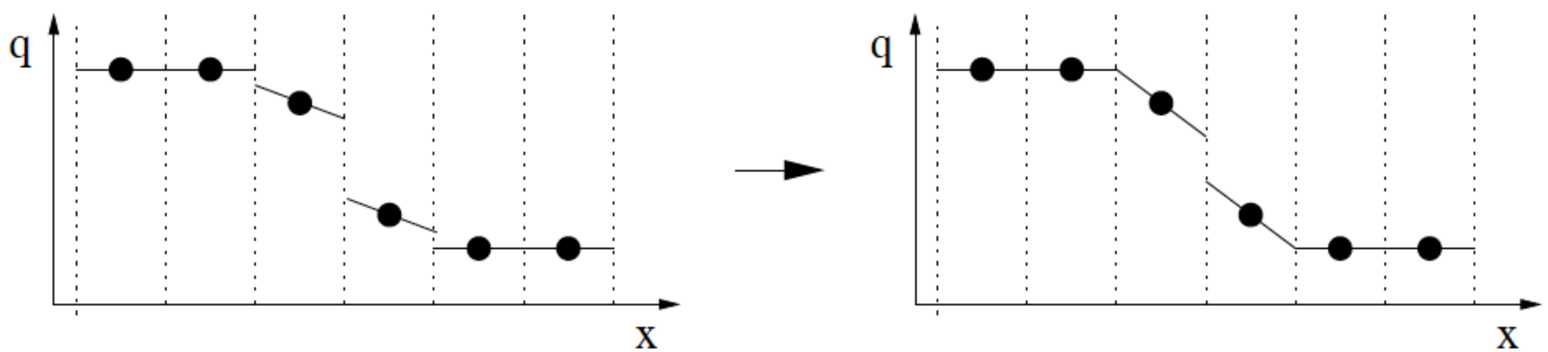


Fix: slope limiters -> flux limiters



minmod slope superbee slope

Flux limiters:

- Normal flux:

$$f_{i+1/2}^{n+1/2} = \begin{cases} u_{i+1/2}\, q_i^n & \text{for } u_{i+1/2} > 0 \\ u_{i+1/2}\, q_{i+1}^n & \text{for } u_{i+1/2} < 0 \end{cases}$$

- Flux correction due to limiter Φ_i

$$\frac{1}{2}\,|u_i| \left(1 - |u_i|\,\frac{\Delta t}{\Delta x}\right) (q_i - q_{i-1})\ \Phi_i$$

Flux-conserving grid-based hydrodynamics

Flux limiters:

- Flux correction due to limiter Φ_i : $\frac{1}{2}|u_i|\left(1-|u_i|\frac{\Delta t}{\Delta x}\right)(q_i - q_{i-1})\,\Phi_i$

$$r^n_{i-1/2} = \begin{cases} \frac{q^n_{i-1}-q^n_{i-2}}{q^n_i - q^n_{i-1}} & \text{for } u_{i-1/2} \geq 0 \\ \frac{q^n_{i+1}-q^n_i}{q^n_i - q^n_{i-1}} & \text{for } u_{i-1/2} \leq 0 \end{cases}$$

$$\begin{aligned}
\text{donor-cell :} \quad & \phi(r) = 0 \\
\text{Lax-Wendroff :} \quad & \phi(r) = 1 \\
\text{Beam-Warming :} \quad & \phi(r) = r \\
\text{Fromm :} \quad & \phi(r) = \tfrac{1}{2}(1+r)
\end{aligned}$$

linear

---

non-linear

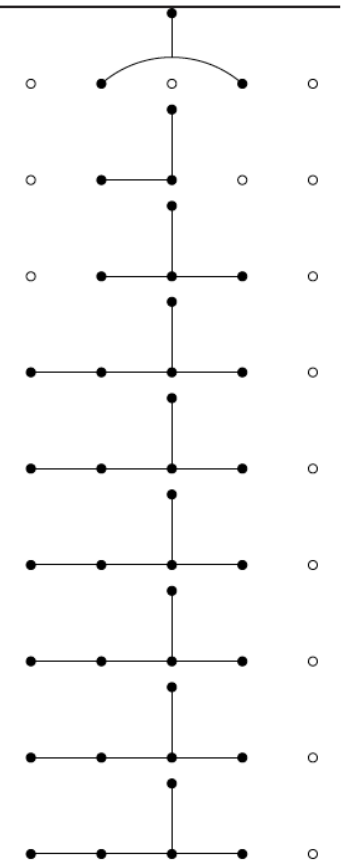
$$\begin{aligned}
\text{minmod :} \quad & \phi(r) = \mathrm{minmod}(1, r) \\
\text{superbee :} \quad & \phi(r) = \max(0, \min(1, 2r), \min(2, r)) \\
\text{MC :} \quad & \phi(r) = \max(0, \min((1+r)/2, 2, 2r)) \\
\text{van Leer :} \quad & \phi(r) = (r + |r|)/(1 + |r|)
\end{aligned}$$

Flux-conserving grid-based hydrodynamics

Flux limiters:

- Flux correction due to limiter Φ_i : $\frac{1}{2}|u_i|\left(1-|u_i|\frac{\Delta t}{\Delta x}\right)(q_i - q_{i-1})\,\Phi_i$

Name	Order	Lin?	Stable?	TVD?	Stencil
Two-point symmetric	1	lin	-	-	
Upwind / Donor-cell	1	lin	+	+	
Lax-Wendroff	2	lin	+	-	
Beam-warming	2	lin	+	-	
Fromm	2	lin	+	-	
Minmod	2/1	non-lin	+	+	
Superbee	2/1	non-lin	+	+	
MC	2/1	non-lin	+	+	
van Leer	2/1	non-lin	+	+	



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construction of classic 1D hydro solver

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \vec{u}) &= \\
\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \vec{u}) &= \\
\partial_t (\rho e_{\rm tot}) + \nabla \cdot (\rho e_{\rm tot} \vec{u}) &=
\end{aligned}
\boxed{
\begin{aligned}
&0 \\
&-\nabla P \\
&-\nabla \cdot (P \vec{u})
\end{aligned}
}
$$

Source terms

HYDRO STEP:

1. Use standard advection scheme to advect $\rho,\ \rho\vec{u},\ \rho e_{\rm tot}$ with zero source

2. Treat source terms separately (operator splitting)

Advantage of operator splitting: source terms cancel exactly (not inside the advection)

construction of classic 1D hydro solver

Code for hydro step; test with interacting sound waves

Building a 2D hydro code in python

1. Advection in 2D
2. Hydro step in 2D
3. Sedov and KH instability in 2D

The basics of grid-based hydrodynamics

1. Advection
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Treating shocks – Riemann solvers

- Code treats smooth flows fairly well

- But shocks are common in astrophysics (e.g., interstellar medium)

- Flow speed is supersonic, i.e., $u > c_s$

- Need to solve Riemann problem

- Leads to Riemann solvers (e.g., Piecewise Parabolic Method)

Collela & Woodward (1984)

Difference to previous solver:
pressure terms are included in the advection

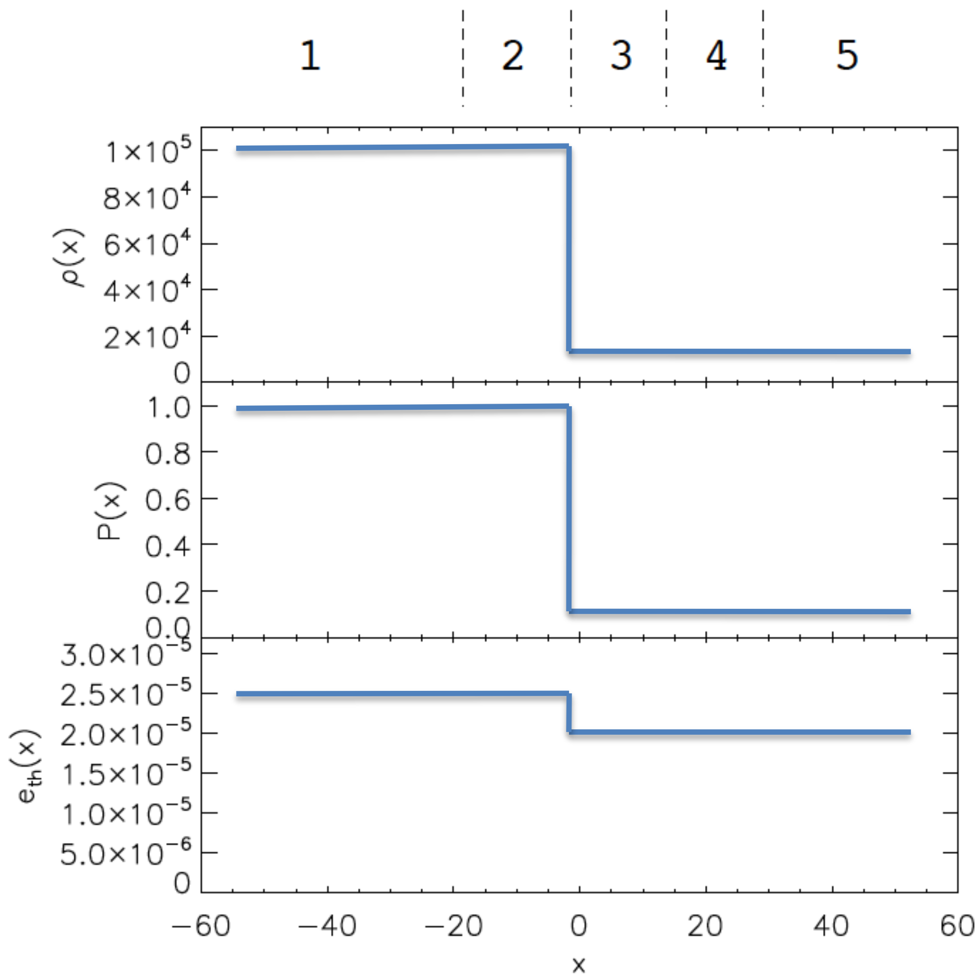
Sod shocktube test: $\rho_l = 10^5, P_l = 1$ $\rho_r = 1.25 \times 10^4$ and $P_r = 0.1$

(Sod 1978)

density

pressure

Internal energy



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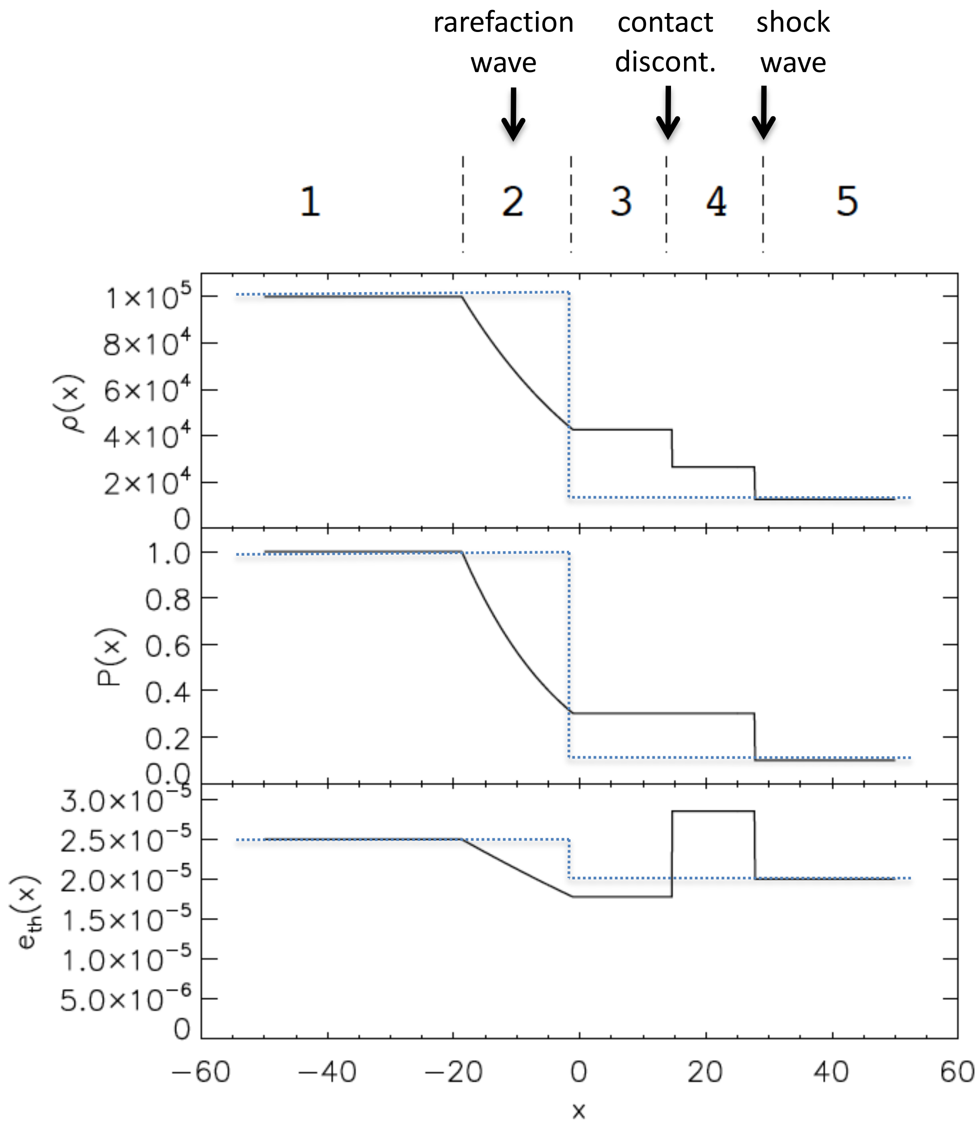

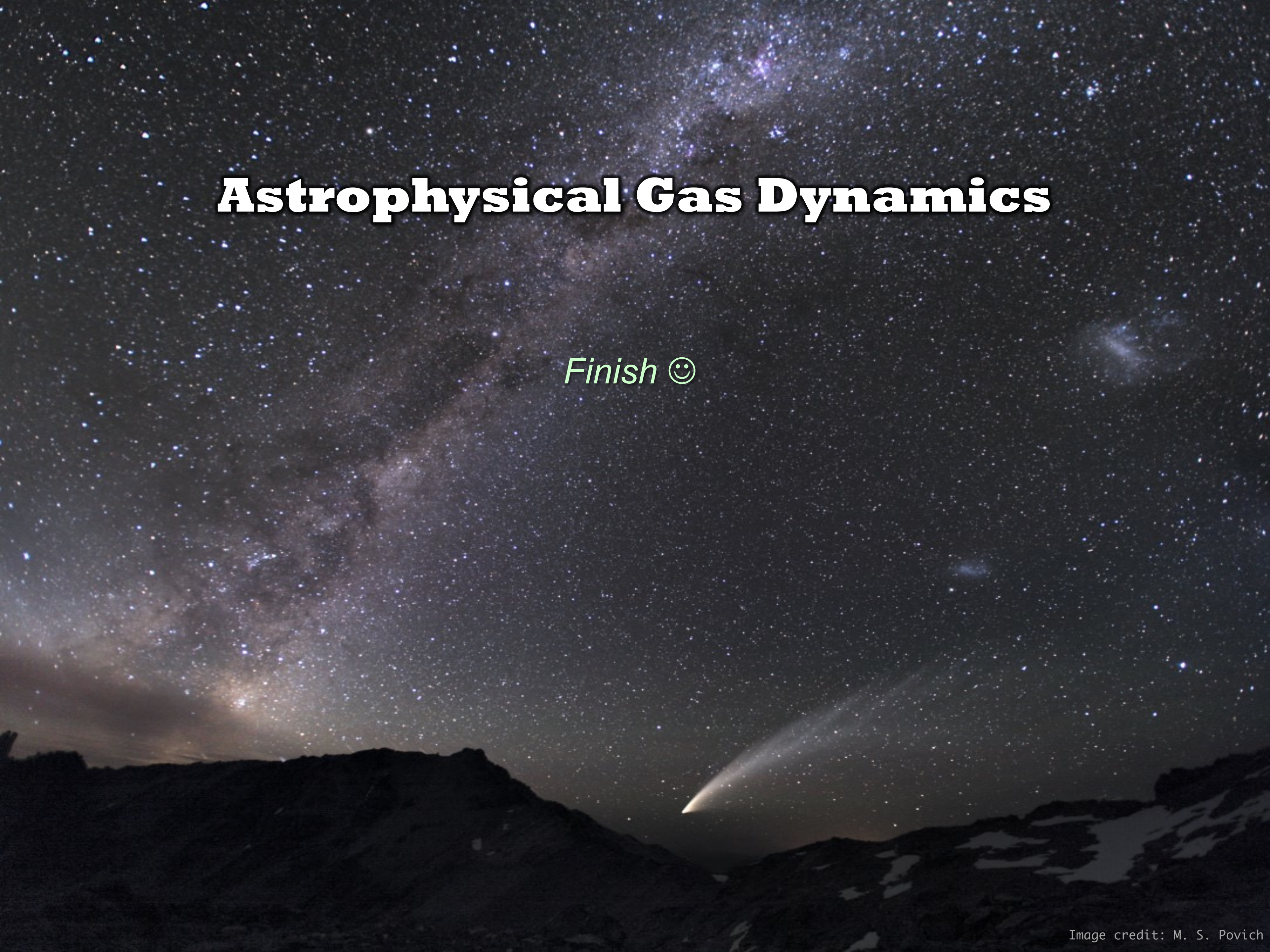
Astrophysical Gas Dynamics
Finish ☺
Image credit: M. S. Povich