

# Wave Solutions to Linearised

## Ideal MHD Equations

Ideal MHD equations are good 1<sup>st</sup> order approximations to molecular clouds  
— the birth place of stars.

Why?

$$\rightarrow Re \sim 10^9 \Rightarrow |U \nabla^2 U| \ll 1$$

$$\rightarrow R_m \sim 10^{16} \Rightarrow |h \nabla^2 B| \ll 1$$

$\Rightarrow$  ideal (w.r.t dissipation) other non-ideal effects may be important

viscous dissipation

magnetic resistivity

also approximately isothermal  $\gtrsim 0.1 \text{ pc}$

$$\text{with } \sigma_v/c_s = \mathcal{M} \sim 10$$

$$L \sim 0(\text{pc}), T = L/v_v \sim 0(\text{Myr})$$

$$\text{Temp.} \sim 10\text{K}, n \sim 100 \text{ cm}^{-3}$$

Hence, let's begin with seeking solutions to the ideal MHD eqs.

## Ideal isothermal MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{continuity}$$

mass advection      thermal pressure

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla P + \underbrace{\frac{1}{4\pi}(\nabla \times \vec{B}) \times \vec{B}}_{\text{momentum}} \quad \text{Lorentz force}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \quad \text{induction}$$

Faraday's + Ampere's + Ohm's law

$$\nabla \cdot \vec{B} = 0 \quad \text{divergence-free constraint} \quad P = c_s^2 \rho \quad \text{EoS.}$$

### Linearised variables:

Consider a plasma with a stationary, homogeneous background with only small (linear) perturbations from the stationary state:

linear-
$\vec{v}_0 \delta X$
$\vec{X}_0^2$
$\vec{X}_0, \delta X$

$$\vec{v} = \vec{v}_0 + \delta \vec{v}(\vec{r}, t) = \delta \vec{v}$$

$$\rho = \rho_0 + \delta \rho(\vec{r}, t)$$

$$\vec{B} = \vec{B}_0 + \delta \vec{B}(\vec{r}, t)$$

Consider  
 $\vec{v}_0 = 0$

no bulk motion

Where  $\frac{\partial_x X_0}{\text{homogenous}} = \frac{\partial_t X_0}{\text{stationary}} = 0$ , where  $X$  is any variable.

Let's also consider  $\beta = P_{\text{turb}} / P_{\text{mag}} \ll 1$   
 appropriate for magnetised molecular clouds.

## Continuity Eq.:

$$\frac{\partial (\rho_0 + \delta\rho)}{\partial t} + \nabla \cdot [(\rho_0 + \delta\rho) \delta v] = 0$$

nonlinear .... bye!

$$\boxed{\frac{\partial \delta\rho}{\partial t} + \nabla \cdot (\rho_0 \delta v) = 0}$$

## Momentum Eq.:

nonlinear ... bye!

$$\begin{aligned} \frac{\partial (\rho_0 + \delta\rho) \delta v}{\partial t} + \nabla \cdot & \underbrace{(\rho_0 + \delta\rho) \delta v \delta v}_{= -\nabla c_s^2 (\rho_0 + \delta\rho) +} \\ & \frac{1}{4\pi} [\nabla \times (B_0 + \delta B)] \times (B_0 + \delta B) \end{aligned}$$

$$\nabla \times (B_0 + \delta B) \times (B_0 + \delta B)$$

$$= [\cancel{\nabla \times B_0}^\circ + \nabla \times \delta B] \times (B_0 + \delta B)$$

$$= \underbrace{(\nabla \times \delta B) \times B_0}_{\text{nonlinear} \sim \delta B^2/L} + (\nabla \times \delta B) \times \delta B$$

$$\boxed{\rho_0 \frac{\partial \delta v}{\partial t} = -c_s^2 \nabla \delta \rho + \frac{1}{4\pi} (\nabla \times \delta B) \times B_0}$$

## Induction Eq.:

$$\frac{\partial (B_0 + \delta B)}{\partial t} = \nabla \times [\delta v \times (B_0 + \delta B)]$$

$$\frac{\partial \delta B}{\partial t} = \nabla \times (\delta v \times B_0)$$

## Linearised MHD Eqs.:

$$\frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_0 \delta v) = 0 \quad \text{lin. continuity}$$

$$\rho_0 \frac{\partial \delta v}{\partial t} = -c_s^2 \nabla \delta \rho + \frac{1}{4\pi} (\nabla \times \delta B) \times B_0 \quad \text{lin. momentum}$$

$$\frac{\partial \delta B}{\partial t} = \nabla \times (\delta v \times B_0) \quad \text{lin. induction}$$

$$\nabla \cdot \delta B = 0 \quad \text{lin divergence.}$$

Solvable with Fourier transform. Let's take the  $\mathcal{F}\{f(x)\} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$  of the

**Reminder:** linearised MHD equations.

$$i) \mathcal{F}\left\{ \frac{\partial^n}{\partial x^n} f(x, t) \right\} = (ik)^n \tilde{f}(k, \omega)$$

$$2) \left\{ \frac{\partial^2}{\partial t^2} f(x, t) \right\} = (-i\omega)^2 \tilde{f}(k, \omega) \quad a, b \in \mathbb{R}$$

$$3) \left\{ af(x, t) + bg(x, t) \right\} = a\tilde{f}(k, \omega) + b\tilde{g}(k, \omega)$$

Fourier transform is linear and turns derivatives into algebra.

Let's apply:

$$-\cancel{i\omega \delta\rho} + \cancel{i\rho_0(k \cdot \tilde{\delta r})} = 0$$

$$-\rho_0 \cancel{i\omega \tilde{\delta r}} = -c_s^2 \cancel{i k \delta\rho} + \frac{1}{4\pi} (\cancel{k \times \tilde{\delta B}}) \times B_0$$

$$-\cancel{i\omega \delta B} = \cancel{i k \times (\tilde{\delta r} \times B_0)}$$

$$\cancel{i k \cdot \delta B} = 0$$

$$-\omega \tilde{\delta\rho} + \rho_0 (k \cdot \tilde{\delta r}) = 0 \quad \text{lin k-space continuity}$$

$$-\rho_0 \omega \tilde{\delta r} = -c_s^2 k \tilde{\delta\rho} + \frac{1}{4\pi} (k \times \tilde{\delta B}) \times B_0 \quad \text{lin k-space momentum}$$

$$-\omega \delta B = k \times (\tilde{\delta r} \times B_0) \quad \text{lin k-space induction}$$

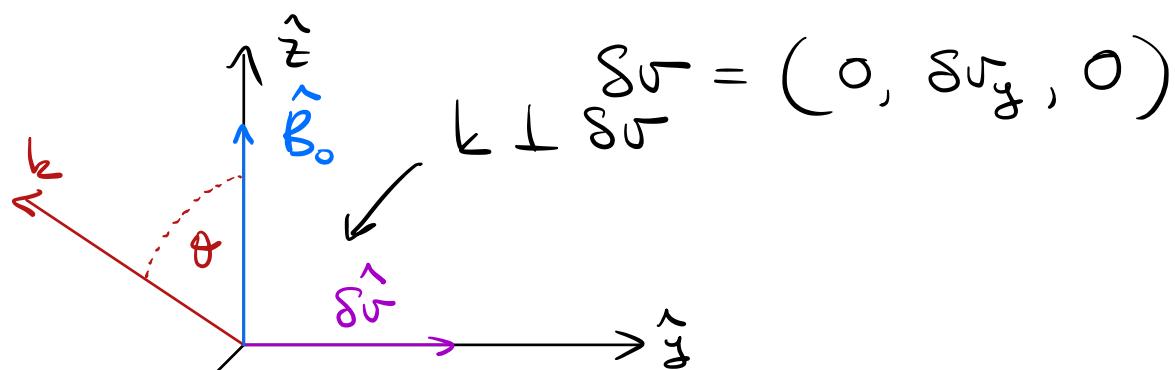
$$k \cdot \delta B = 0 \quad \text{lin. k-space divergence-free.}$$

Choose wave geometry:

let  $\mathbf{k} = (k_x, 0, k_z)$ ,  $\hat{\mathbf{B}}_0 = B_0 \hat{\mathbf{z}}$

This is just for pedagogy but always have freedom to pick coordinates.

Transverse waves:



$$\mathbf{k} \perp \delta \mathbf{v} = (0, \delta v_y, 0)$$

Density Perturbation:

$$-\omega \tilde{\delta \rho} + \rho_0 (\mathbf{k} \cdot \tilde{\delta \mathbf{v}}) = 0$$
$$\tilde{\delta \rho} = \rho_0 \frac{\mathbf{k} \cdot \tilde{\delta \mathbf{v}}}{\omega}$$

$$\text{Since } \mathbf{k} \perp \delta \mathbf{v} \Rightarrow \mathbf{k} \cdot \tilde{\delta \mathbf{v}} = 0$$

$$\Rightarrow \tilde{\delta \rho} = 0$$

i.e. Transverse waves do not perturb density.  $\Rightarrow$  Incompressible.

Velocity and Magnetic Perturbations

linearised momentum:

Only y-component survives

$$-\rho_0 \omega \tilde{\delta v}_y = -c_s^2 k \tilde{\delta p} + \frac{1}{4\pi} (\mathbf{k} \times \tilde{\delta \mathbf{B}}) \times \mathbf{B}_0$$

$$\rho_0 \omega \tilde{\delta v}_y = \frac{1}{4\pi} (\mathbf{k} \times \tilde{\delta \mathbf{B}}) \times \mathbf{B}_0$$

$$\rho_0 \omega \tilde{\delta v}_y + \frac{k_z B_0}{4\pi} \tilde{\delta B}_y = 0$$

linearised induction:

only  $y$ -component survives

$$-\omega \tilde{\delta B}_y = \mathbf{k} \times (\tilde{\delta v} \times \mathbf{B}_0)$$

$$k_z B_0 \tilde{\delta v}_y + \omega \tilde{\delta B}_y = 0$$

Combine linearised equations:

$$\begin{bmatrix} \rho_0 \omega & \frac{k_z B_0}{4\pi} \\ k_z B_0 & \omega \end{bmatrix} \begin{bmatrix} \tilde{\delta v}_y \\ \tilde{\delta B}_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only interested in non-trivial solutions  
 $\tilde{\delta v}_y \neq 0, \tilde{\delta B}_y \neq 0$ .

$$\Rightarrow \det \left( \begin{bmatrix} \rho_0 \omega & \frac{k_z B_0}{4\pi} \\ k_z B_0 & \omega \end{bmatrix} \right) = 0$$

Reminder

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

$$\text{Hence: } \rho_0 \omega^2 - \frac{k_z^2 B_0^2}{4\pi} = 0$$

$$\omega^2 = k_z^2 \left( \frac{B_0^2}{4\pi \rho_0} \right)$$

$$\boxed{\omega^2 = k_z^2 V_{A0}^2}$$

where  $V_{A0} = B_0 / \sqrt{4\pi \rho_0}$  is the Alfvén speed.

*dispersion relation for Alfvén waves*

The full relation is:

$$\Rightarrow k_z = k \cos \theta$$

|  $\theta$  is angle between  $\hat{k}$  and  $\hat{B}_0$

$$\boxed{\omega^2 = k^2 V_{A0}^2 \cos^2 \theta}$$

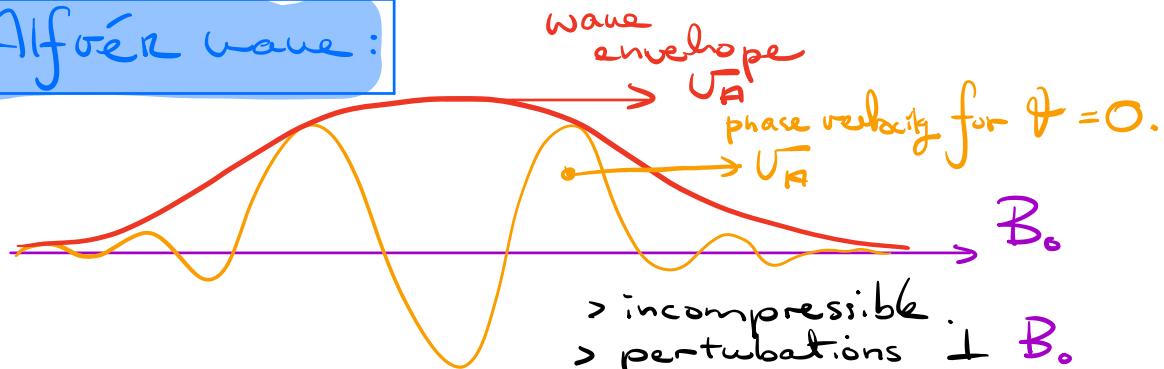
for planar waves of the form  $A(x, t) = A_0 e^{i(kx - \omega t)}$

$$\frac{d\omega}{dk} := \text{group velocity} = V_{A0} \cos \theta \quad \text{independent of } k$$

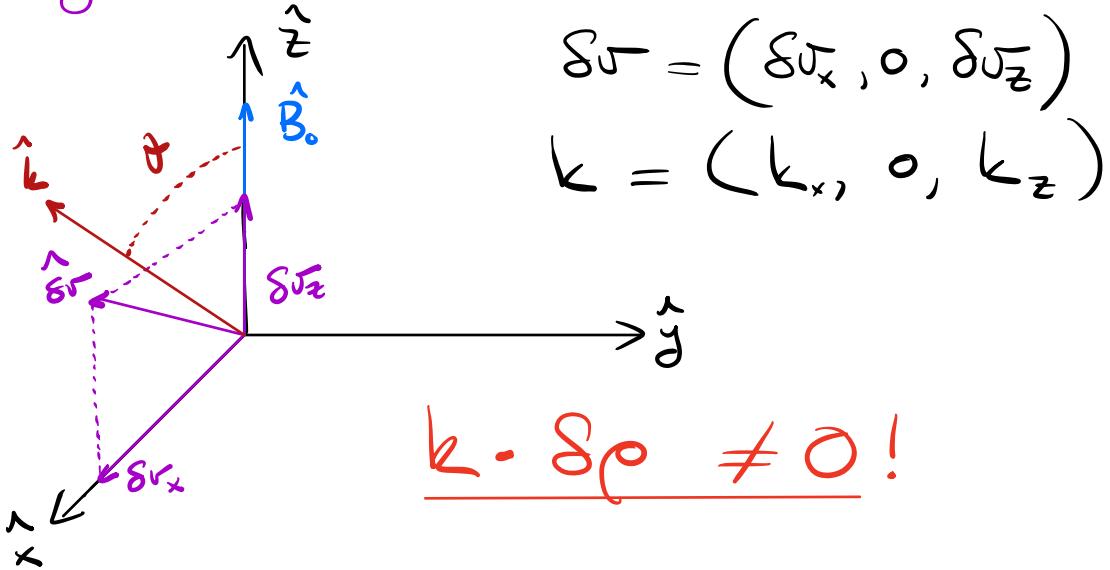
$$\frac{\omega}{k} := \text{phase velocity} = V_{A0} \cos \theta$$

What we are describing is an

**Alfvén wave:**



## Longitudinal Waves



## Perturbations

$$\begin{aligned}
 \cancel{\delta B} - \omega \tilde{\delta B} &= \vec{k} \times (\tilde{\delta v} \times \vec{B}_0) \quad \text{lin } k\text{-space} \\
 &= \vec{k} \times (-\tilde{\delta v}_x \vec{B}_0) \hat{y} \\
 \tilde{\delta B} &= \frac{\tilde{\delta v}_x \vec{B}_0}{\omega} (\vec{k} \times \hat{y}) \\
 \cancel{\delta v}, \cancel{\delta B} / \dots \text{ leave for the reader}
 \end{aligned}$$

## Dispersion relation:

Same as previously but for  $\tilde{\delta \rho}$ ,  $\tilde{\delta B}$ ,  $\tilde{\delta v}$ . This is because  $\vec{k} \cdot \tilde{\delta v} \neq 0$ .

This results in a  $3 \times 3$  matrix problem of the form  $\det(A) = 0$ , with

characteristic equation :

$$\omega^4 - \omega^2 k^2 (V_{A0}^2 + C_s^2) + k^2 k_z^2 V_A^2 C_s^2 = 0$$

which is a quadratic in  $\omega^2 = ?$

$$?^2 - k^2 (V_{A0}^2 + C_s^2) ? + k^4 \cos^2 \theta V_{A0}^2 C_s^2 = 0$$

$$\omega_{1,2}^2 = ?_1,2 = \frac{k^2}{2} V_{\text{fast}}^2 \pm \sqrt{\frac{1}{4} k^4 V_{\text{fast}}^4 - k^4 V_{A0}^2 C_s^2 \cos^2 \theta}$$

$$\text{where } V_{\text{fast}}^2 = V_{A0}^2 + C_s^2.$$

Simplify and separate roots

$$\begin{aligned} \omega_{1,2}^2 &= k^2 \left( \frac{1}{2} V_{\text{fast}}^2 \pm \sqrt{\frac{V_{\text{fast}}^4}{4} - V_{A0}^2 C_s^2 \cos^2 \theta} \right) \\ &= \frac{k^2}{2} V_{\text{fast}}^2 \left( 1 \pm 2 \sqrt{\frac{1}{4} - \frac{V_{A0}^2 C_s^2}{V_{\text{fast}}^4} \cos^2 \theta} \right) \\ &= \frac{k^2}{2} V_{\text{fast}}^2 \left[ 1 \pm \sqrt{1 - 4 \frac{V_{A0}^2 C_s^2}{V_{\text{fast}}^4} \cos^2 \theta} \right] \end{aligned}$$

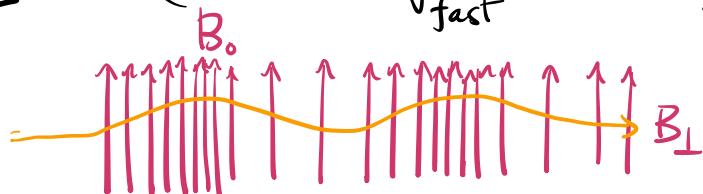
Use binomial approximation

$$(1 + x)^\alpha \approx 1 + \alpha x$$

$$f \rightarrow \left| \frac{V_{A0}^2 C_s^2}{(V_{A0}^2 + C_s^2)^2} \right| \ll 1, \chi = -4 \frac{\sqrt{V_{A0}^2 C_s^2}}{V_{\text{fast}}} \cos^2 \theta$$

$$\omega_{1,2}^2 \simeq \frac{k^2}{2} V_{\text{fast}}^{-2} \left[ 1 \pm \left( 1 - 2 \frac{\sqrt{V_{A0}^2 C_s^2}}{V_{\text{fast}}} \cos^2 \theta \right) \right]$$

**Positive root**



$$\omega^2 \simeq k^2 V_{\text{fast}}^{-2} - \frac{\sqrt{V_{A0}^2 C_s^2}}{V_{\text{fast}}} \frac{\cos^2 \theta}{\cos^2 \theta}$$

since  $\left| \frac{V_{A0}^2 C_s^2}{V_{\text{fast}}^4} \right| \ll 1$  for  $\beta \ll 1$  plasma dispersion

$$\omega^2 \simeq k^2 V_{\text{fast}}^{-2}$$

relation for fast magnetosonic wave.

> compressible,  $k \cdot \nabla v \neq 0$ .

> maximised at  $V_{\text{fast}} \perp B_0$ .

group velocity = phase velocity =  $V_{\text{fast}}$

**negative root**

$$\omega^2 \simeq \frac{k^2}{2} V_{\text{fast}}^{-2} \left[ 1 - \left( 1 - 2 \frac{\sqrt{V_{A0}^2 C_s^2}}{V_{\text{fast}}} \cos^2 \theta \right) \right]$$

$$\approx k^2 \frac{V_{A0}^2 C_s^2}{V_{A0}^2 + C_s^2} \cos^2 \theta$$

$$\omega^2 \approx k^2 \left( C_s^2 - \frac{C_s^4}{V_{A0}^2 + C_s^2} \right) \cos^2 \theta$$

$|C_s^4 / (V_{A0}^2 + C_s^2)| \ll 1$  because  $V_{A0} \gg C_s$   
for  $\beta \ll 1$ .

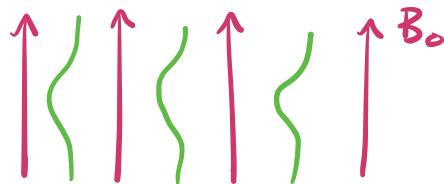
$$\omega^2 \approx k^2 C_s^2 \cos^2 \theta$$

dispersion  
relation  
for slow wave.

> compressible.

> travels at  $C_s \parallel B_0$ .

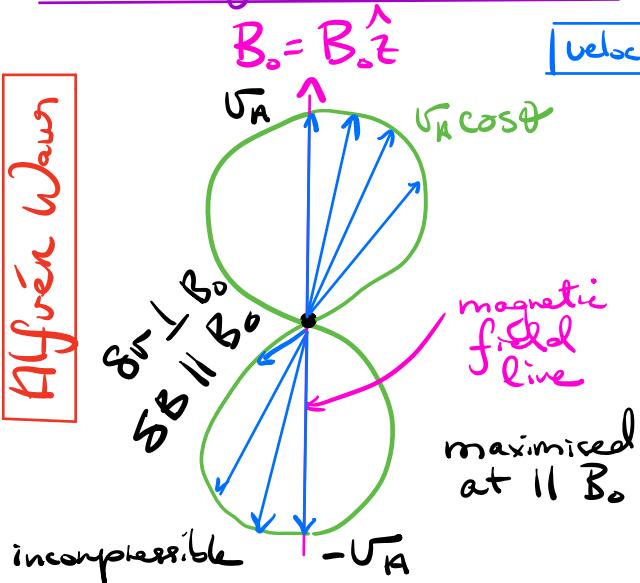
> do not travel  $\perp B_0$ .



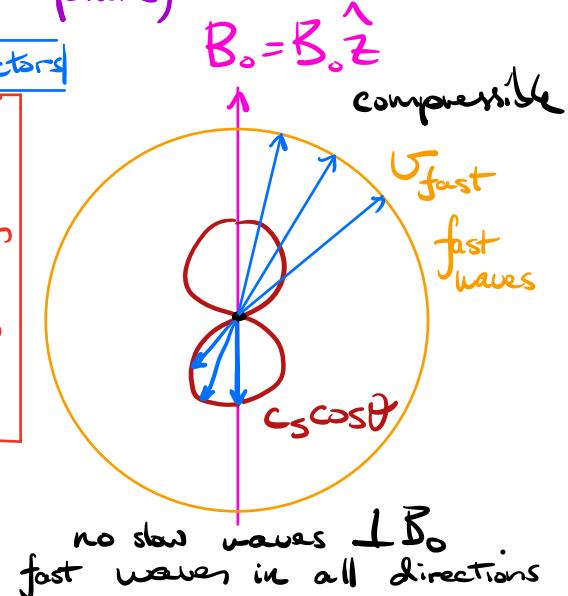
group velocity = phase velocity =  $C_s \cos \theta$

### Summary of 3-waves ( $B_0$ -v plane)

Afgrén Waves



Velocity vectors



Note :

- 1) linearised MHD equations not good for supersonic turbulence.  
e.g.  $|\delta \rho^2| \ll 1$  to be ignored. However  
 $(\delta \rho) \sim M^2 \Rightarrow |\delta \rho^2| \sim M^4 \Rightarrow$   
non-linear terms matter.

- 2) MHD turbulence has been historically framed as a wave-interaction phenomena between Alfvén wave packets.

Fundamentally this ignores other waves, however we're learning of their importance in the last few years.