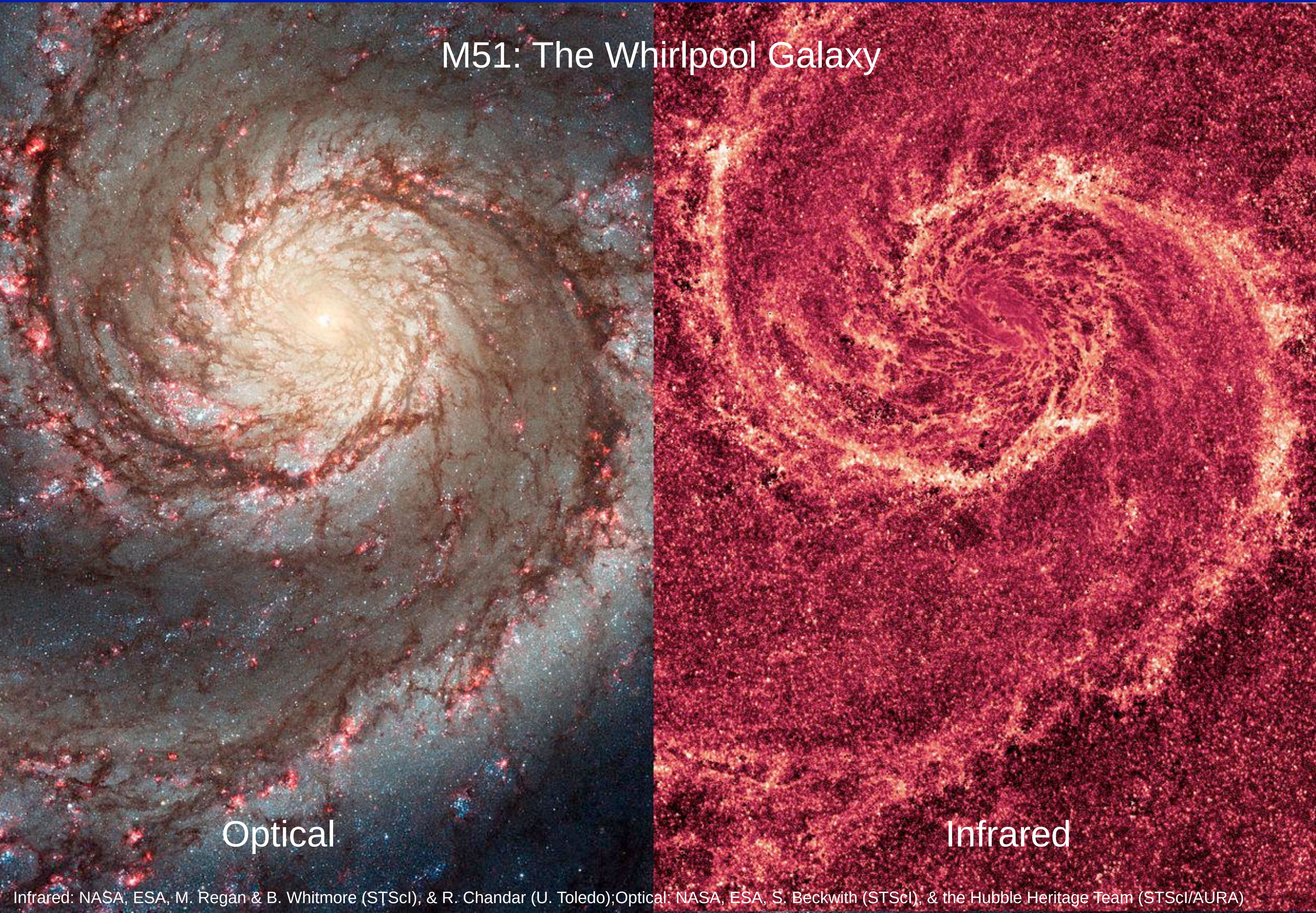
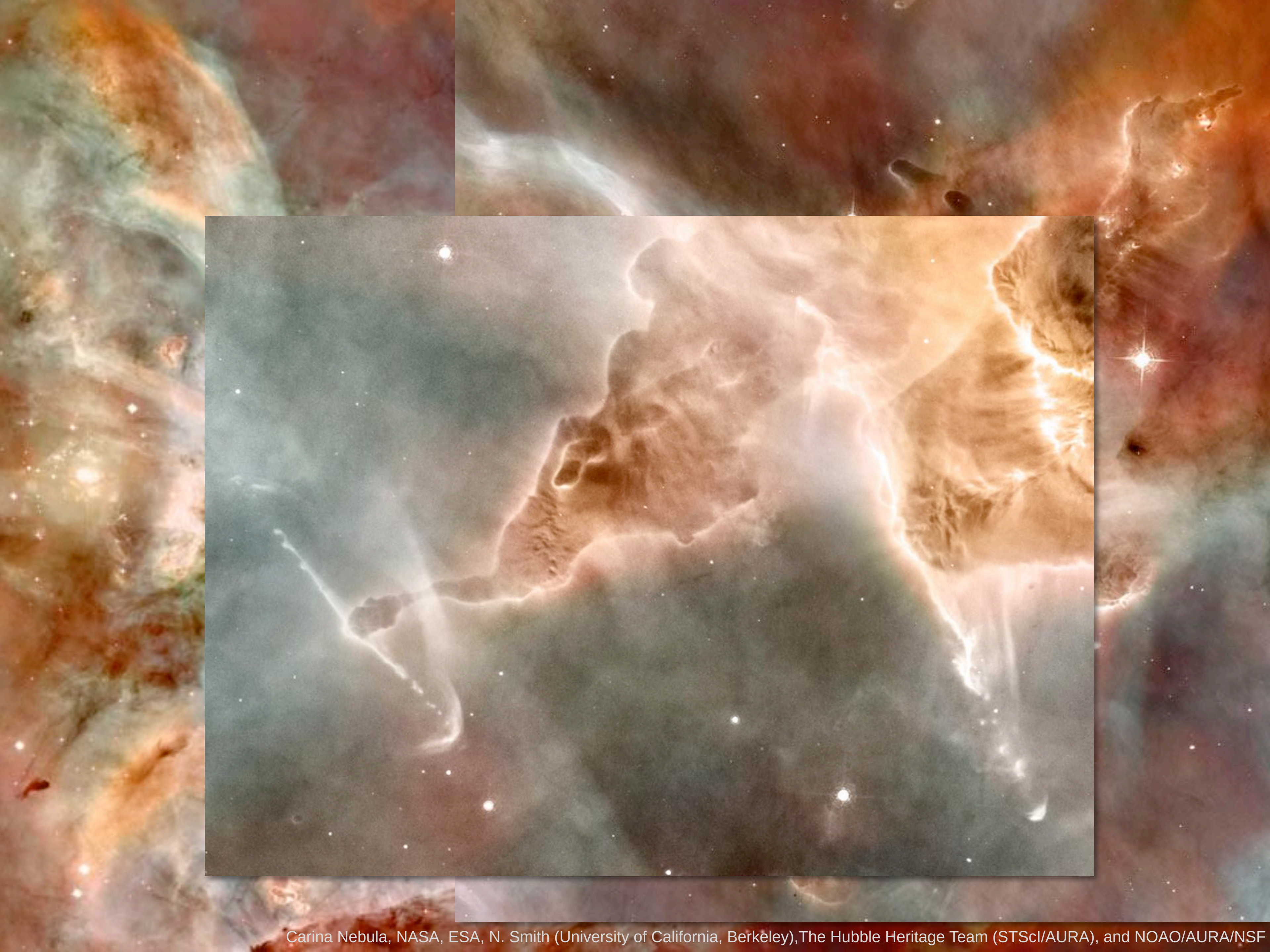
Grid-based Hydrodynamics

Christoph Federrath

- Complex fluid dynamics (equations are non-linear, 3D)
- Complex physics: turbulence, gravity, radiation, magnetic fields, etc.
- Large spatial and temporal scales involved

Star Formation

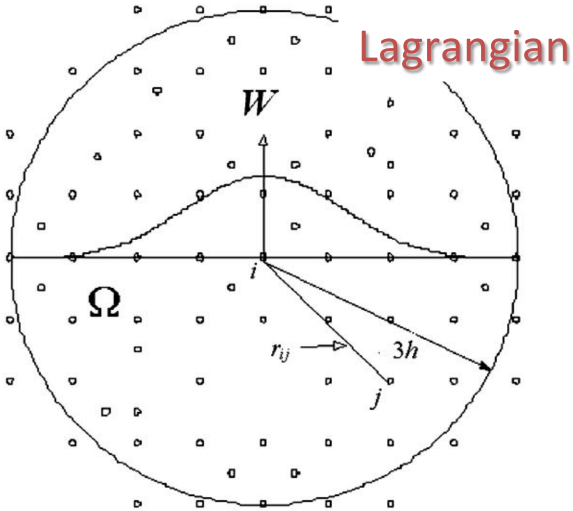




Carina Nebula, NASA, ESA, N. Smith (University of California, Berkeley),The Hubble Heritage Team (STScI/AURA), and NOAO/AURA/NSF

Fundamentals of SPH and grid

Smoothed Particle Hydrodynamics (SPH)
(Lucy 1977; Gingold & Monaghan 1977)

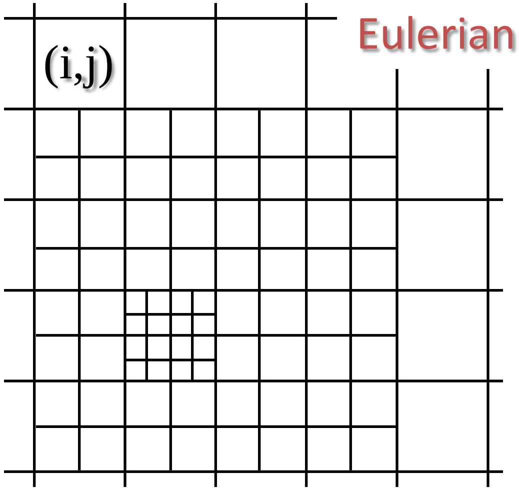


$$\rho(\mathbf{r}) = \sum_b m_b W(\mathbf{r} - \mathbf{r}_b, h)$$

$$\nabla A(\mathbf{r}) = \sum_b m_b \frac{A_b}{\rho_b} \nabla W(\mathbf{r} - \mathbf{r}_b, h)$$

$$W(x,h) = \frac{1}{h\sqrt{\pi}} e^{-(x^2/h^2)}$$

Adaptive Mesh Refinement (AMR)
(Berger & Collela 1989)



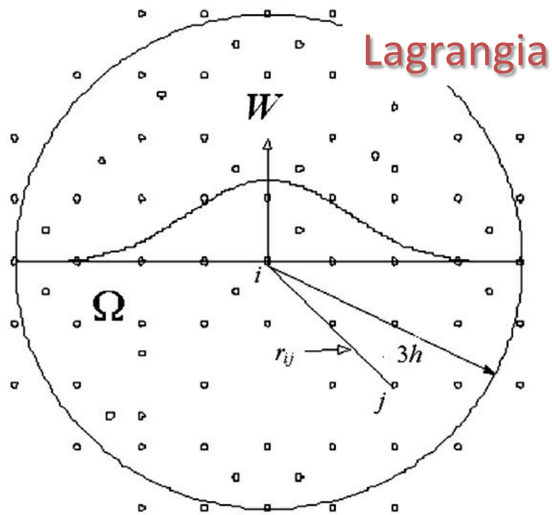
- Hydro variables are averages in cells
- Compute fluxes through cell faces
- Simple data structure: indexing
- Finite Volume vs. Finite Difference

R. Leveque: „Nonlinear Conservation Laws and Finite Volume Methods for Astrophysical Fluid Flow“

Fundamentals of SPH and grid

Smoothed Particle Hydrodynamics (SPH)
(Lucy 1977; Gingold & Monaghan 1977)

Lagrangian



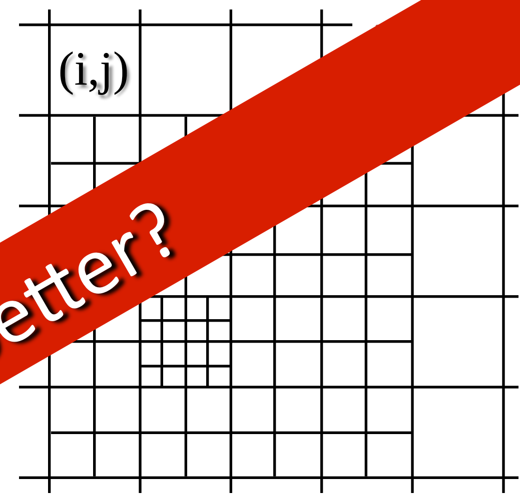
$\rho(\mathbf{r}) = \sum_b m_b W(\mathbf{r} - \mathbf{r}$ ...

$\nabla A(\mathbf{r})$ ... $\nabla W(\mathbf{r} - \mathbf{r}_b, h)$ ... ρ_b

W ... $, h) = \frac{1}{h\sqrt{\pi}} e^{-(x^2/h^2)}$

Which method is better?

Adaptive Mesh Refinement (AMR)
(Berger & Collela 1989)



- Hydro variables are averages in cells
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R. Leveque: „Nonlinear Conservation Laws and Finite Volume Methods for Astrophysical Fluid Flow“

Comparison of SPH and grid in supersonic turbulence

TWO REGIMES OF TURBULENT FRAGMENTATION AND THE STELLAR INITIAL MASS FUNCTION FROM PRIMORDIAL TO PRESENT-DAY STAR FORMATION

Paolo Padoan,[1] Åke Nordlund,[2] Alexei G. Kritsuk,[1] Michael L. Norman,[1] and Pak Shing Li[3]
Received 2006 October 16; accepted 2007 February 16

Their conclusion:

" SPH simulations of large scale star formation to date fail in all three fronts: numerical diffusivity, numerical resolution, and presence of magnetic fields. This should cast serious doubts on the value of comparing predictions based on SPH simulations with observational data (see also Agertz et al. 2006). "

Driven turbulence comparison of SPH and grid

Motivation (role of supersonic turbulence for star formation)

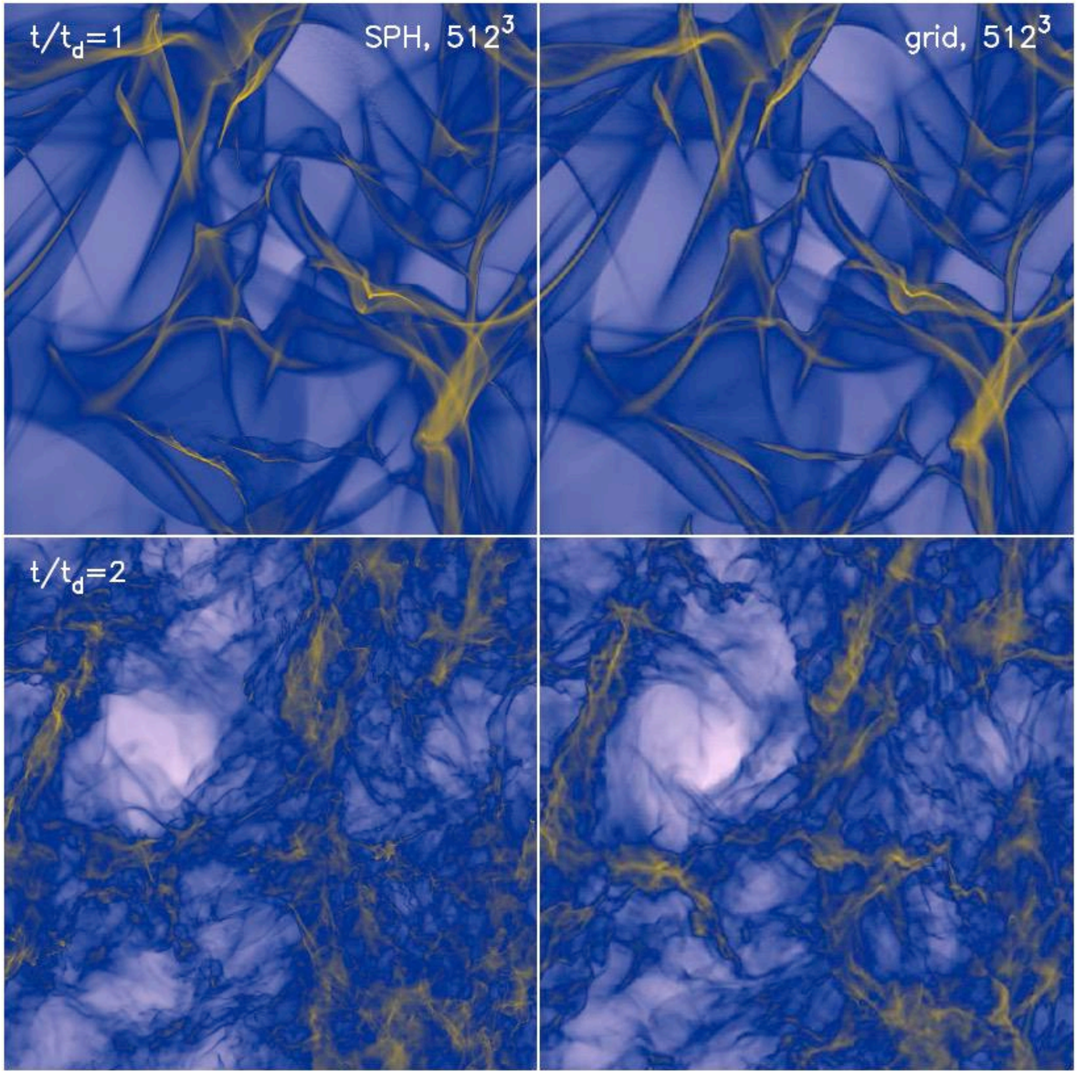
Setup (Phantom and FLASH):

1. Same initial conditions: uniform density, zero velocities
2. Same turbulence forcing!
3. Driven to Mach number 10
4. Resolutions: 128^3, 256^3 and 512^3 (134,217,728) both grid and SPH

(Price & Federrath 2010, MNRAS 406, 1659)

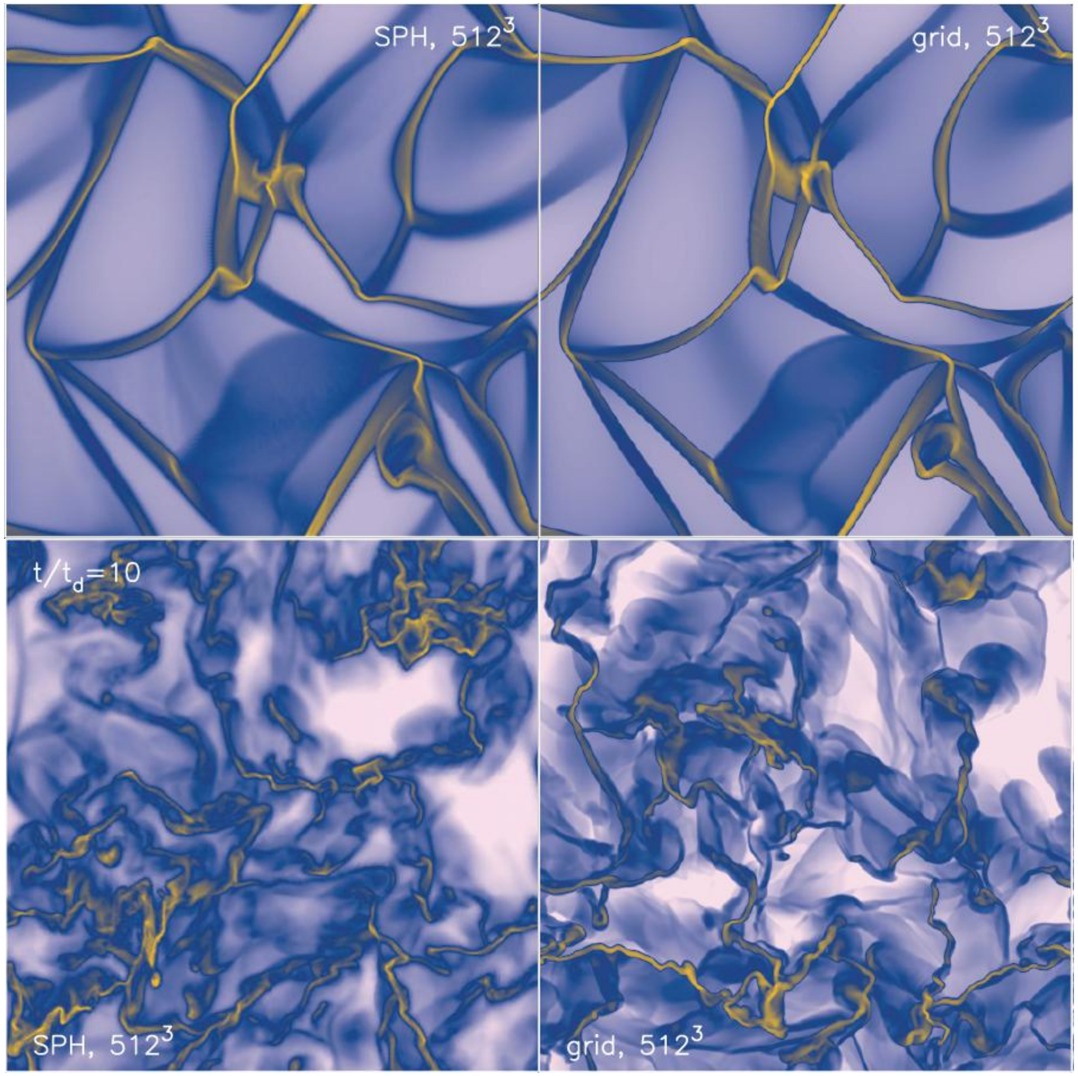
Driven turbulence comparison of SPH and grid

Phantom — Column density — FLASH



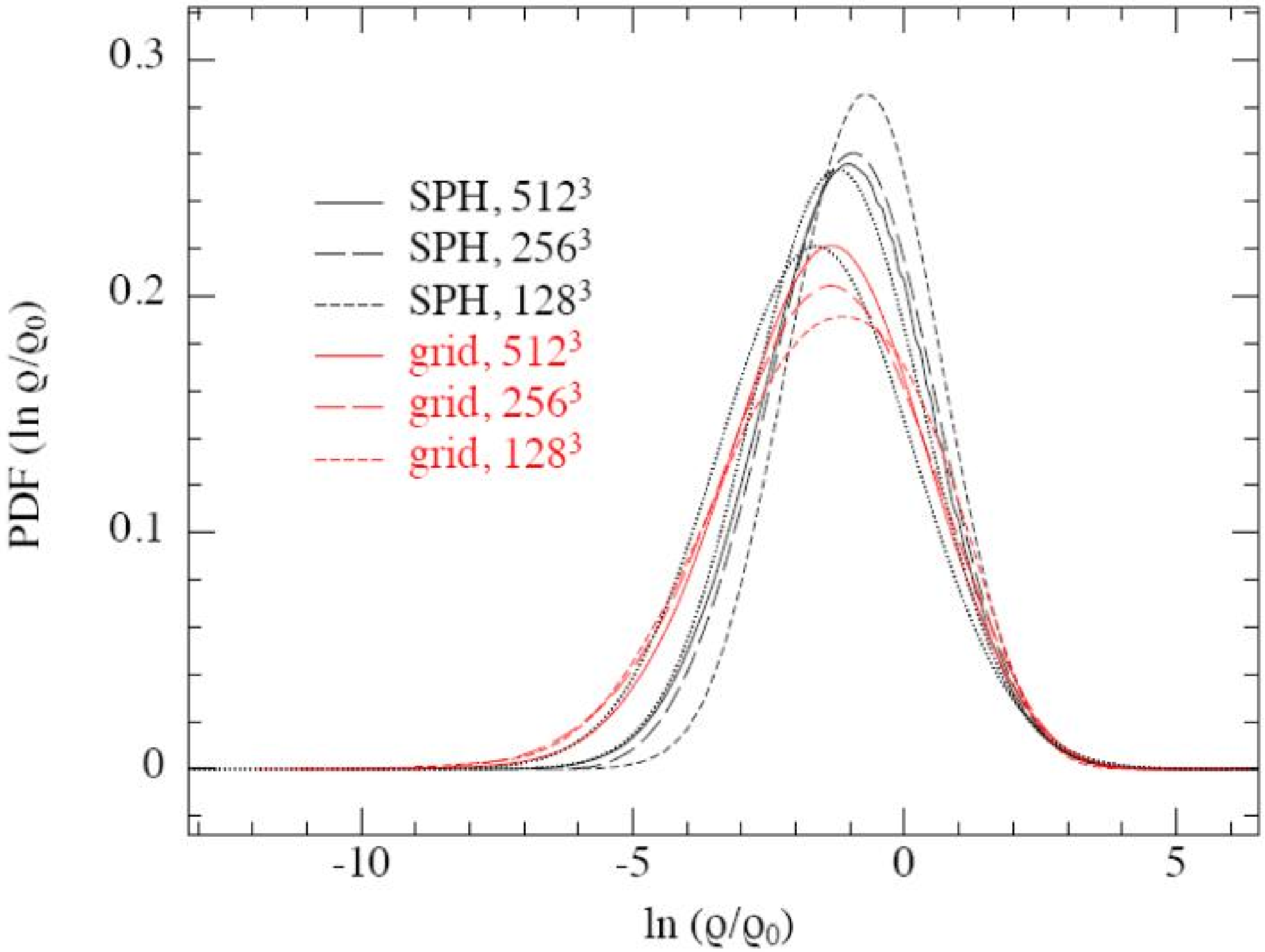
Driven turbulence comparison of SPH and grid

Phantom | Slices of density | FLASH



Driven turbulence comparison of SPH and grid

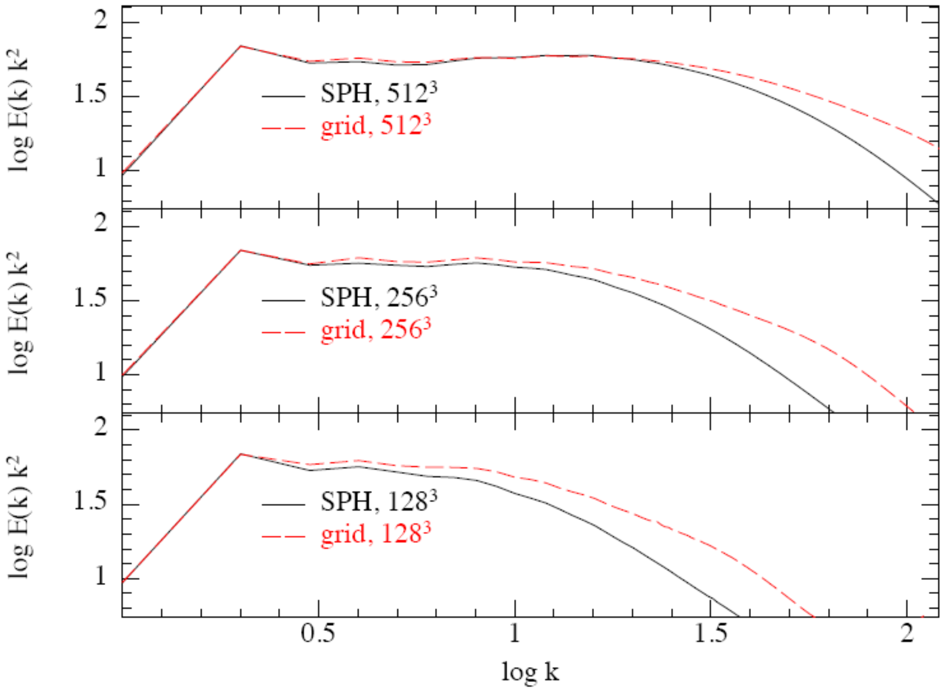
Density Probability Distribution Function (PDF):



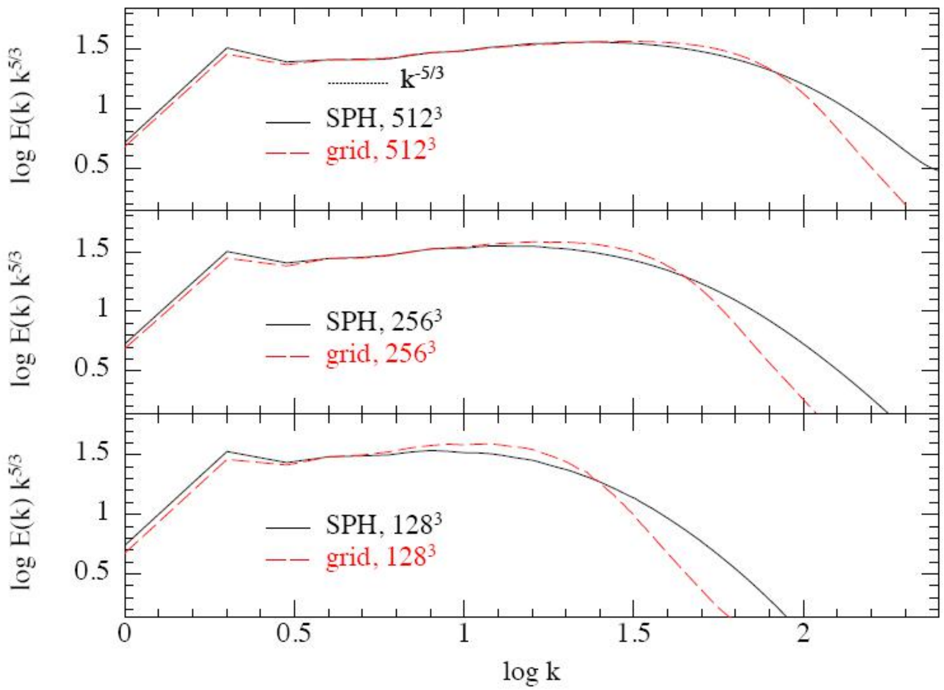
PDFs converge with higher resolution

Driven turbulence comparison of SPH and grid

Velocity spectra, v
(VOLUME-weighted)



Velocity spectra, $\rho^{1/3}v$
(DENSITY-weighted)

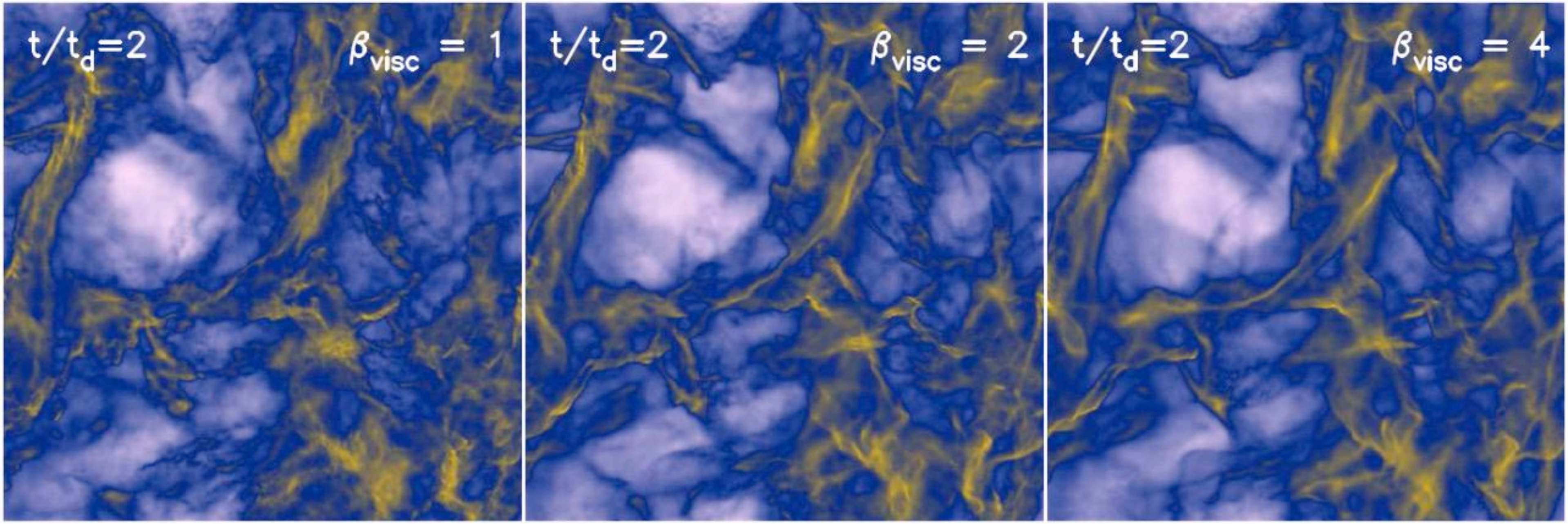


Grid code less dissipative

SPH code slightly less dissipative

Driven turbulence comparison of SPH and grid

Influence of β-viscosity in SPH on the modelling of strong shocks



β=1 β=2 β=4

Particle interpenetration for β<4

Driven turbulence comparison of SPH and grid

Conclusion

(Price & Federrath 2010, MNRAS 406, 1659)

Convergence of SPH and grid

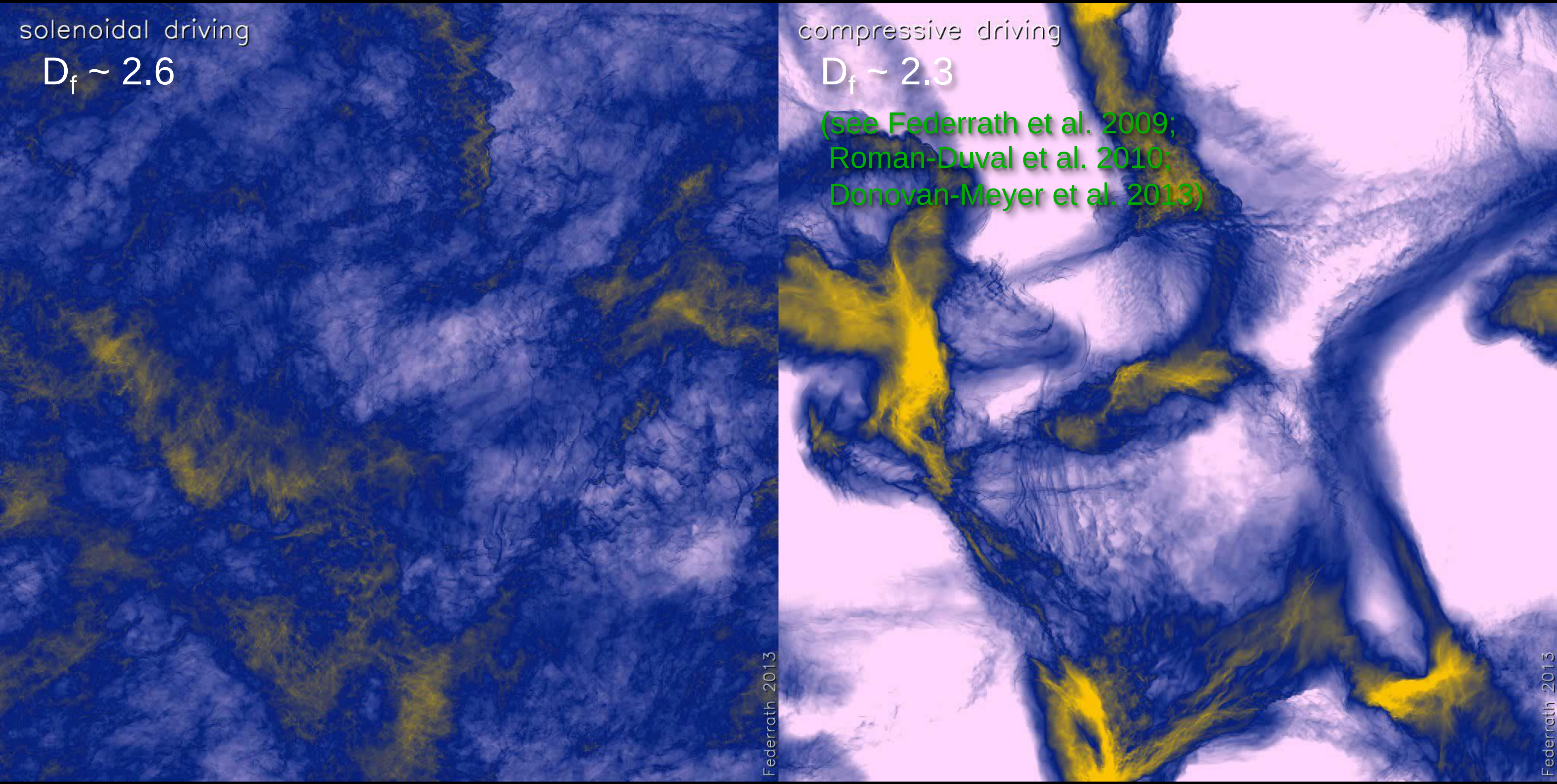
Computational time pure hydro (no gravity):

FLASH grid about 20 times faster than Phantom SPH

Hydrodynamical Turbulence

Movies available: http://www.mso.anu.edu.au/~chfeder/pubs/supersonic/supersonic.html

World's largest simulations of turbulence using 4096^3 grid cells



(Federrath 2013, MNRAS 436, 1245: Supersonic turbulence @ 4096^3 grid cells)

The basics of grid-based hydrodynamics

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6. Basics of Riemann problem -> Riemann solvers
7. Adaptive-mesh refinement and sink particles

Lecture based on a lecture given by Kees Dullemond, 2009/2010, Heidelberg

Literature: Randall J. LeVeque, ”Finite Volume Methods for Hyperbolic Problems”
(Cambridge Texts in Applied Mathematics)

Advection test, IDL code

Flux-conserving grid-based hydrodynamics

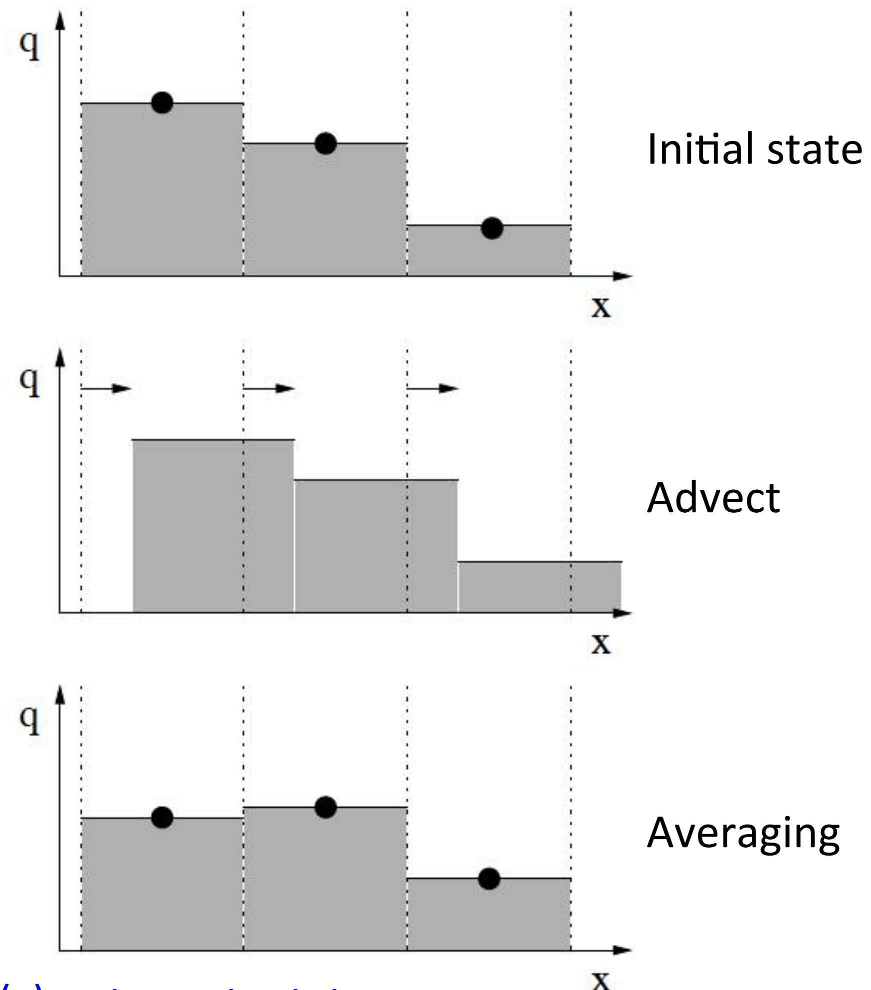
Donor-cell advection:

Piecewise constant subgrid model:

$$\tilde{q}_{i+1/2}^{n+1/2} = \begin{cases} q_i^n & \text{for } u_{i+1/2} > 0 \\ q_{i+1}^n & \text{for } u_{i+1/2} < 0 \end{cases}$$

Flux:

$$f_{i+1/2}^{n+1/2} = \begin{cases} u_{i+1/2}\, q_i^n & \text{for } u_{i+1/2} > 0 \\ u_{i+1/2}\, q_{i+1}^n & \text{for } u_{i+1/2} < 0 \end{cases}$$



Like upwind scheme, but works for u(x) not constant, too.

Flux-conserving grid-based hydrodynamics

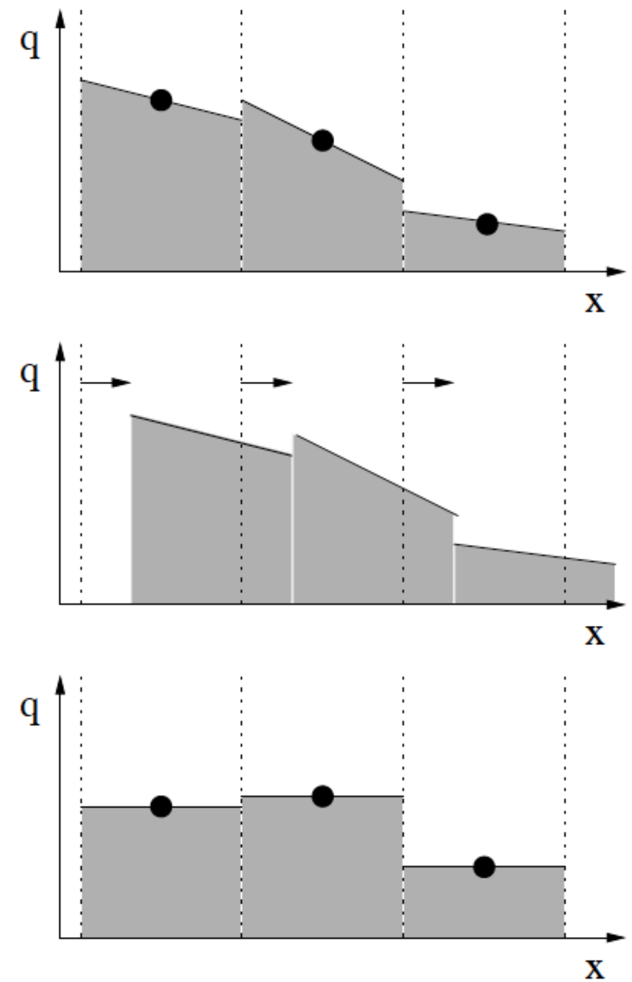
Piecewise linear subgrid model for flux:

- Donor-cell is quite diffusive ->
  Use higher-order subgrid model

$$q(x, t = t_n) = q_i^n + \sigma_i^n (x - x_i)$$

(slope)

Choice of slope



„MUSCL (Monotonic Upwind-centered Scheme for Consveration Laws)“

Flux-conserving grid-based hydrodynamics

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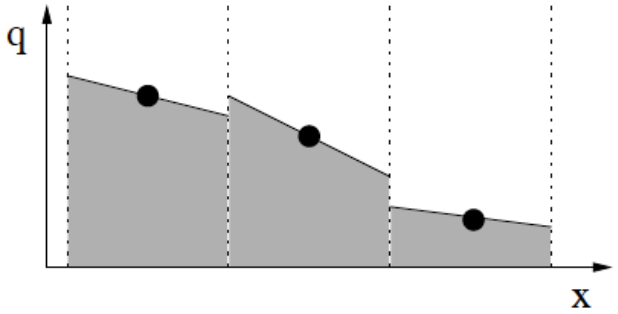


$$q(x, t = t_n) = q_i^n + \sigma_i^n (x - x_i)$$

Different slope choices:

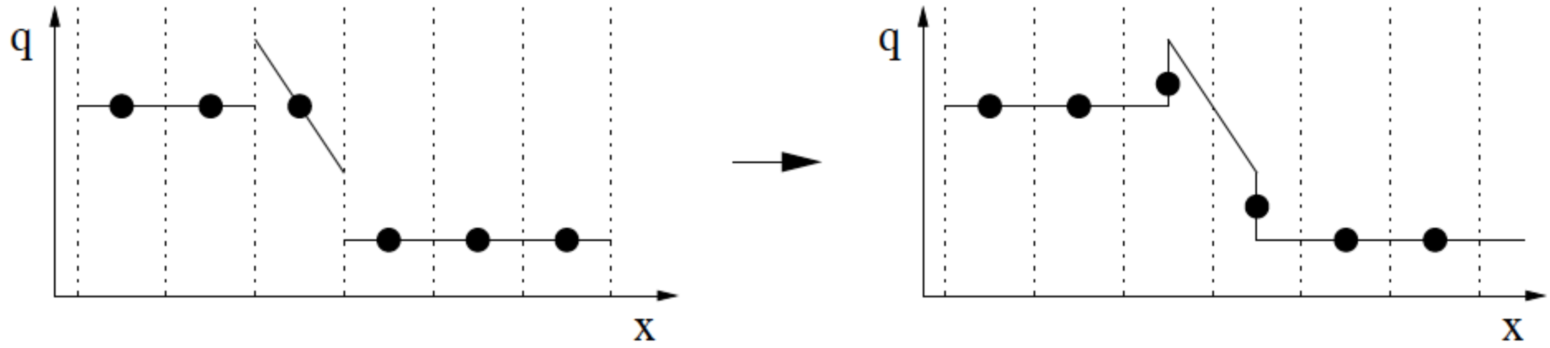
Centered slope:	$\sigma_i^n = \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$	(Fromm's method)
Upwind slope:	$\sigma_i^n = \frac{q_i^n - q_{i-1}^n}{\Delta x}$	(Beam-Warming method)
Downwind slope:	$\sigma_i^n = \frac{q_{i+1}^n - q_i^n}{\Delta x}$	(Lax-Wendroff method)

Higher-order now, but beware oscillations

„MUSCL (Monotonic Upwind-centered Scheme for Consveration Laws)“

Piecewise linear subgrid model for flux:

- can produce overshoots



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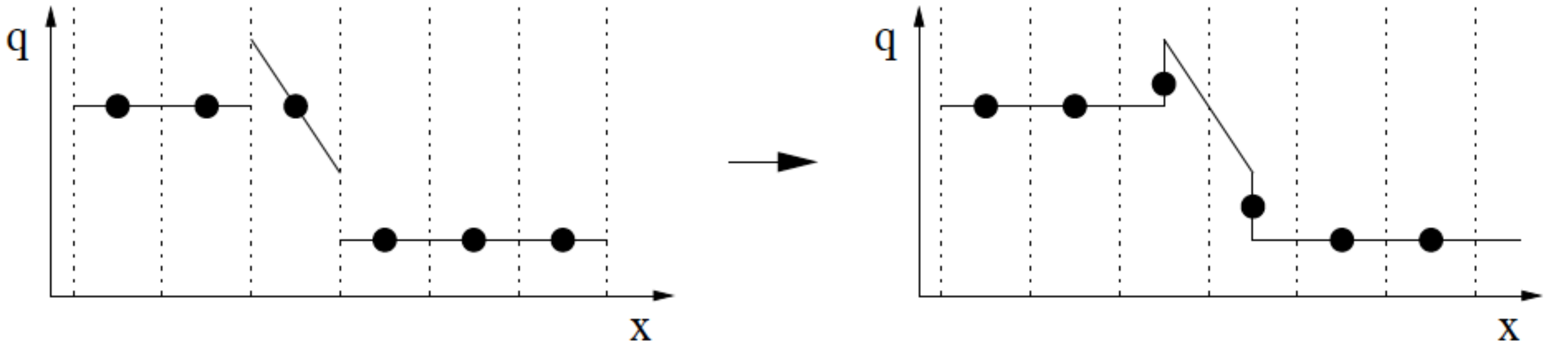
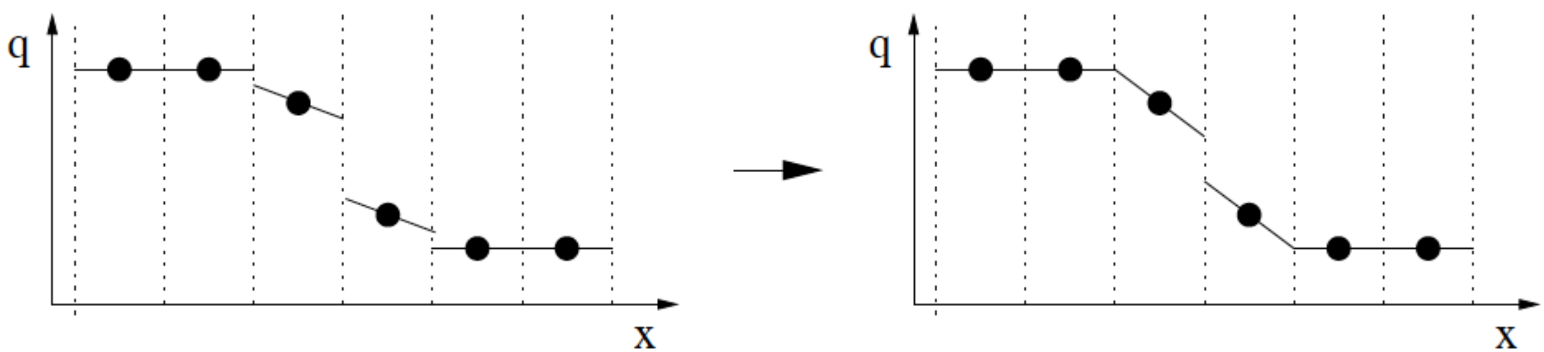


Fix: slope limiters -> flux limiters



minmod slope

superbee slope

Flux-conserving grid-based hydrodynamics

Flux limiters:

- Normal flux:

$$f_{i+1/2}^{n+1/2} = \begin{cases} u_{i+1/2}\, q_i^n & \text{for } u_{i+1/2} > 0 \\ u_{i+1/2}\, q_{i+1}^n & \text{for } u_{i+1/2} < 0 \end{cases}$$

- Flux correction due to limiter Φ_i

$$\frac{1}{2}\,|u_i|\left(1 - |u_i|\,\frac{\Delta t}{\Delta x}\right)(q_i - q_{i-1})\,\Phi_i$$

Flux-conserving grid-based hydrodynamics

Flux limiters:

- Flux correction due to limiter Φ_i : $\frac{1}{2}|u_i|\left(1-|u_i|\frac{\Delta t}{\Delta x}\right)(q_i - q_{i-1})\,\Phi_i$

$$r^n_{i-1/2} = \begin{cases} \frac{q^n_{i-1}-q^n_{i-2}}{q^n_i-q^n_{i-1}} & \text{for } u_{i-1/2} \geq 0 \\ \frac{q^n_{i+1}-q^n_i}{q^n_i-q^n_{i-1}} & \text{for } u_{i-1/2} \leq 0 \end{cases}$$

donor-cell : $\phi(r) = 0$

Lax-Wendroff : $\phi(r) = 1$

Beam-Warming : $\phi(r) = r$

Fromm : $\phi(r) = \frac{1}{2}(1+r)$

linear

---

non-linear

minmod : $\phi(r) = \mathrm{minmod}(1, r)$

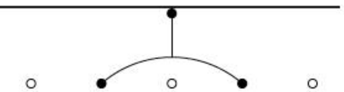
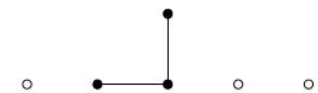
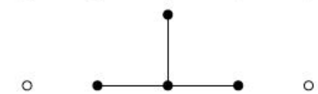
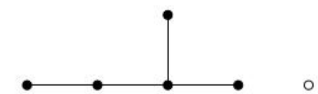
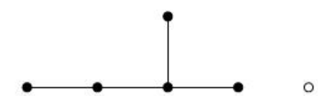
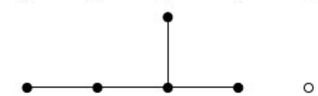
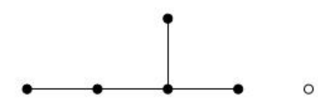
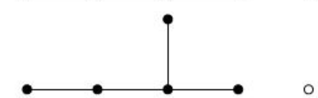
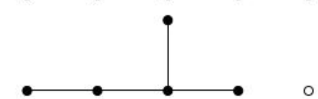
superbee : $\phi(r) = \max(0, \min(1, 2r), \min(2, r))$

MC : $\phi(r) = \max(0, \min((1+r)/2, 2, 2r))$

van Leer : $\phi(r) = (r + |r|)/(1 + |r|)$

Flux limiters:

- Flux correction due to limiter Φ_i : $\frac{1}{2}|u_i|\left(1-|u_i|\frac{\Delta t}{\Delta x}\right)(q_i - q_{i-1})\,\Phi_i$

Name	Order	Lin?	Stable?	TVD?	Stencil
Two-point symmetric	1	lin	-	-	
Upwind / Donor-cell	1	lin	+	+	
Lax-Wendroff	2	lin	+	-	
Beam-warming	2	lin	+	-	
Fromm	2	lin	+	-	
Minmod	2/1	non-lin	+	+	
Superbee	2/1	non-lin	+	+	
MC	2/1	non-lin	+	+	
van Leer	2/1	non-lin	+	+	

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construction of classic 1D hydro solver

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \\
\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \vec{u}) &= -\nabla P \\
\partial_t (\rho e_{\mathrm{tot}}) + \nabla \cdot (\rho e_{\mathrm{tot}} \vec{u}) &= -\nabla \cdot (P \vec{u})
\end{aligned}
$$

Source terms

HYDRO STEP:

1. Use standard advection scheme to advect $\rho,\ \rho\vec{u},\ \rho e_{\mathrm{tot}}$ with zero source

2. Treat source terms separately (operator splitting)

Advantage of operator splitting: source terms cancel exactly (not inside the advection)

construction of classic 1D hydro solver

Code for hydro step; test with interacting sound waves

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Treating shocks – Riemann solvers

- Code treats smooth flows fairly well
- But shocks are common in astrophysics (e.g., interstellar medium)
- Flow speed is supersonic, i.e., $u > c_s$
- Need to solve Riemann problem
- Leads to Riemann solvers (e.g., Piecewise Parabolic Method)

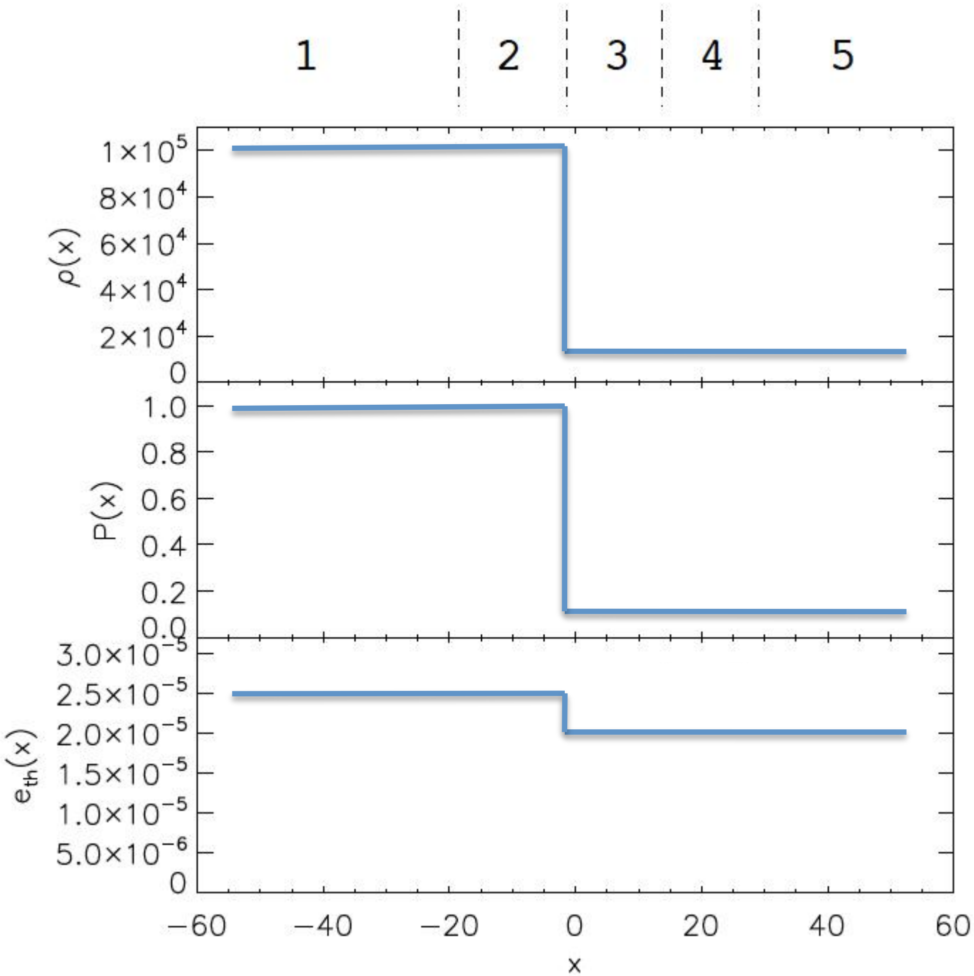
Collela & Woodward (1984)

Difference to previous solver:
pressure terms are included in the advection

Treating shocks

Sod shocktube test: $\rho_l = 10^5, P_l = 1$ $\rho_r = 1.25 \times 10^4$ and $P_r = 0.1$

(Sod 1978)



Sod shocktube test: $\rho_l = 10^5,\, P_l = 1 \qquad \rho_r = 1.25 \times 10^4 \text{ and } P_r = 0.1$

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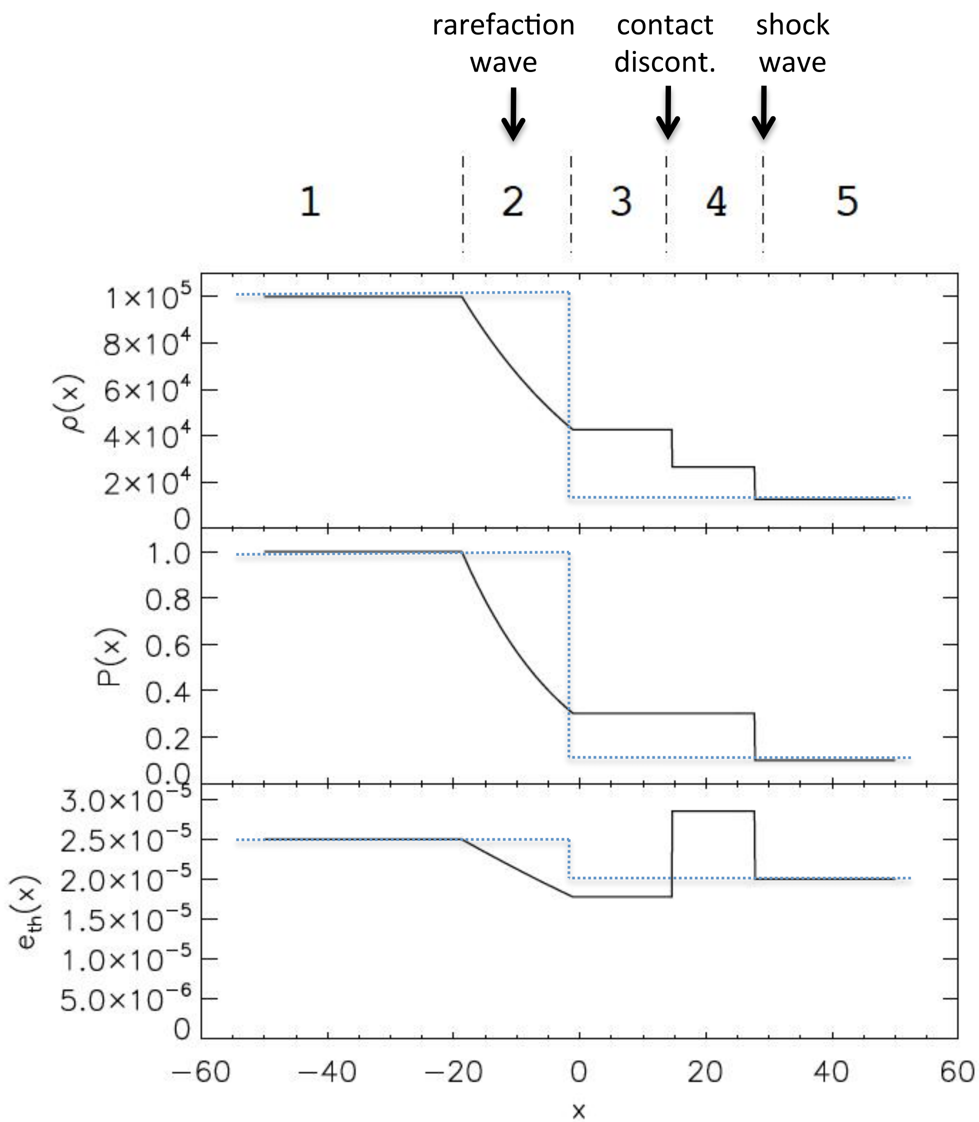

Treating shocks

Sod shocktube test in 1D and 2D with AMR

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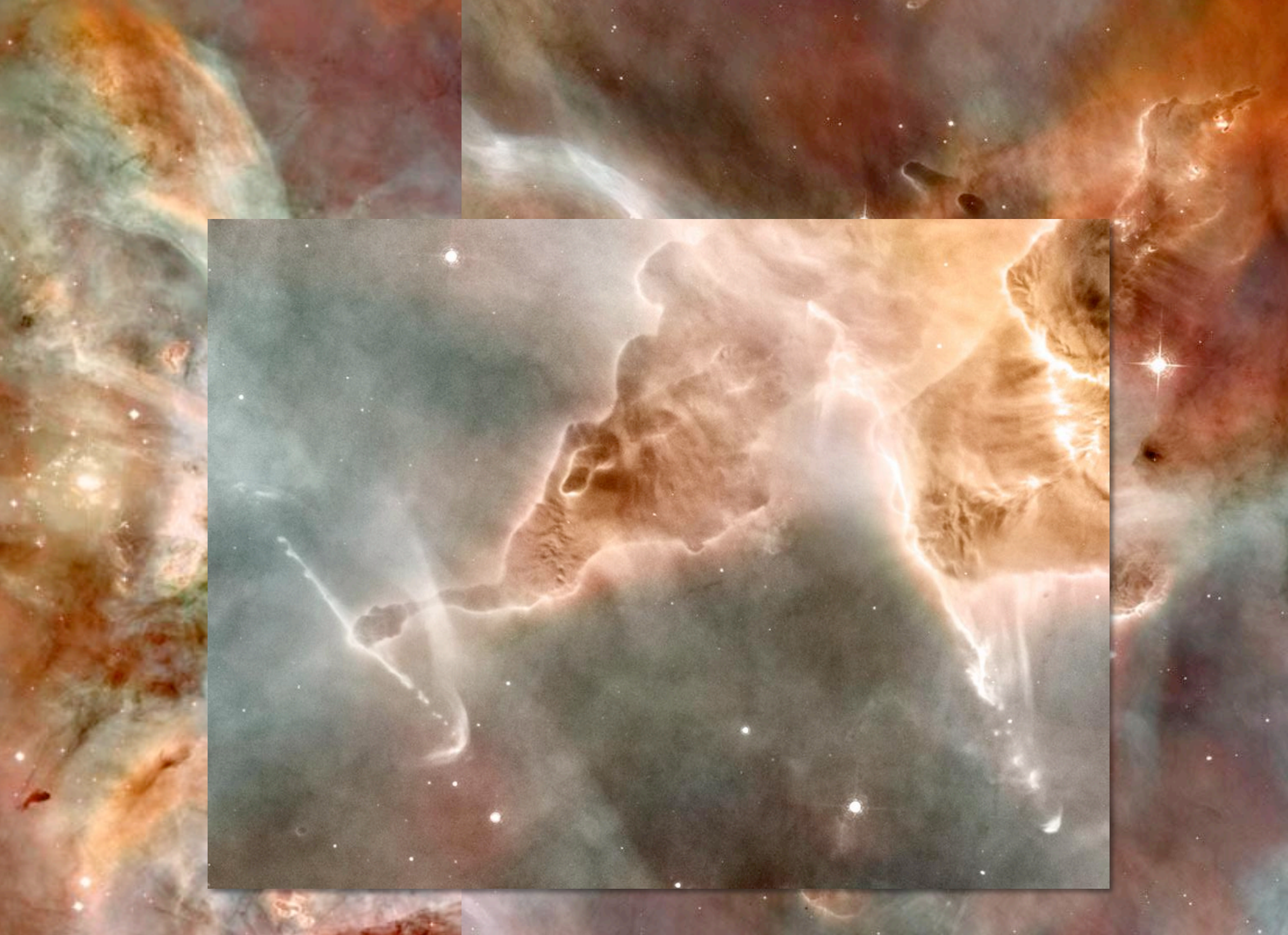

Carina Nebula, NASA, ESA, N. Smith (University of California, Berkeley),The Hubble Heritage Team (STScI/AURA), and NOAO/AURA/NSF

Why sink particles?

- Quantify fragmentation and accretion
- Prevent code from stalling

$$t_{\mathrm{ff}} = \left(\frac{3\pi}{32 G \rho} \right)^{1/2}$$

Why sink particles?

- Quantify fragmentation and accretion
- Prevent code from stalling

$$t_{\rm ff} = \left(\frac{3\pi}{32 G \rho} \right)^{1/2}$$

- Resolve fragmentation scale

$$\lambda_{\rm J} = \left(\frac{\pi c_{\rm s}^2}{G\rho} \right)^{1/2}$$

$$M_{\rm J}(\rho) = \frac{4\pi}{3} \left(\frac{\lambda_{\rm J}(\rho)}{2} \right)^3 \rho$$

Truelove et al. (1997)

Bate & Burkert (1997)

Why sink particles?

- Quantify fragmentation and accretion
- Prevent code from stalling

$$t_{\mathrm{ff}} = \left(\frac{\ldots}{32 G\rho} \right)^{1/2}$$

- Resolve fragmentati… e

Cut off runaway collapse

$$\lambda \ldots \left(\frac{\ldots_{\mathrm{s}}}{G\rho} \right)^{1/2}$$

$$M_{\mathrm{J}}(\rho) = \frac{4\pi}{3} \left(\frac{\lambda_{\mathrm{J}}(\rho)}{2} \right)^3 \rho$$

Truelove et al. (1997)

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Cut off runaway collapse

$$\rho > \rho_{\rm res}$$

Sink Particles

Heat up gas

Cut off runaway collapse

$$\rho \ > \ \rho_{\rm res}$$

Sink Particles

Heat up gas

1. **Problem**:
   Courant time step
   $$\min_{i,j,k}\left(\frac{\Delta x}{\max(|\mathbf{v}(i,\ j,\ k)|,\ c_{\rm s})}\right)$$
2. **Problem**:
   changes EOS,
   unless
   $$\rho_{\rm res} > 10^{-14}\,{\rm g\,cm^{-3}}$$

Cut off runaway collapse

$$\rho \;>\; \rho_{\mathrm{res}}$$

Sink Particles

Problem:

Spurious sink creation in shocks that DON'T go into free fall collapse

e.g., isothermal shock:

(Density~Mach^2)

Heat up gas

1. **Problem**:

   Courant time step

$$\min_{i,j,k}\left(\frac{\Delta x}{\max(|\mathbf{v}(i,\,j,\,k)|,\,c_{\mathrm{s}})}\right)$$

2. **Problem**:

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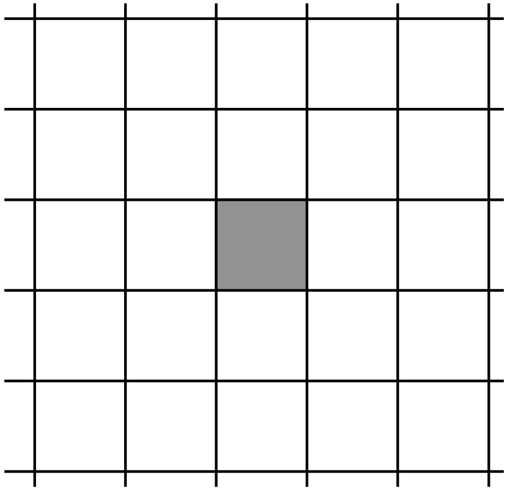
$$\rho_{\mathrm{res}} > 10^{-14}\,\mathrm{g\,cm^{-3}}$$

Collapse checks to avoid spurious sink creation

1. Cell exceeds density threshold, $\rho > \rho_{\mathrm{res}}$



Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Sink particle implementation in FLASH

Collapse checks to avoid spurious sink creation

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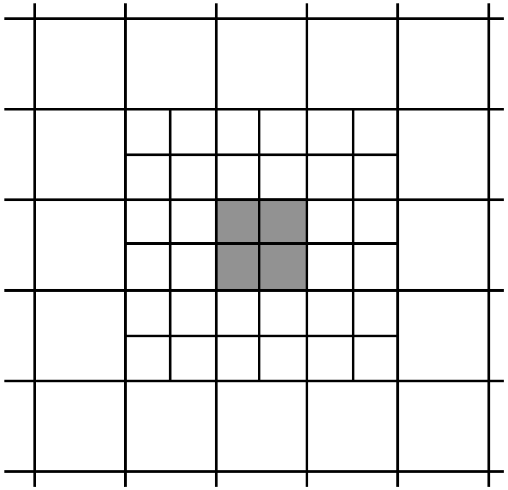

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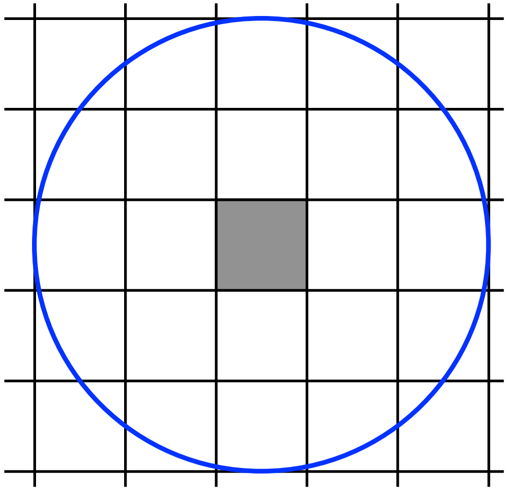

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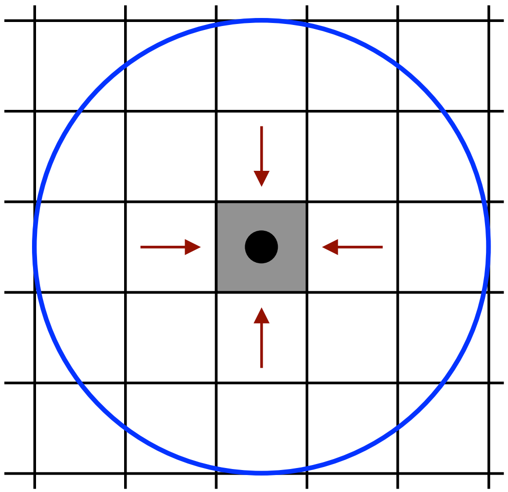
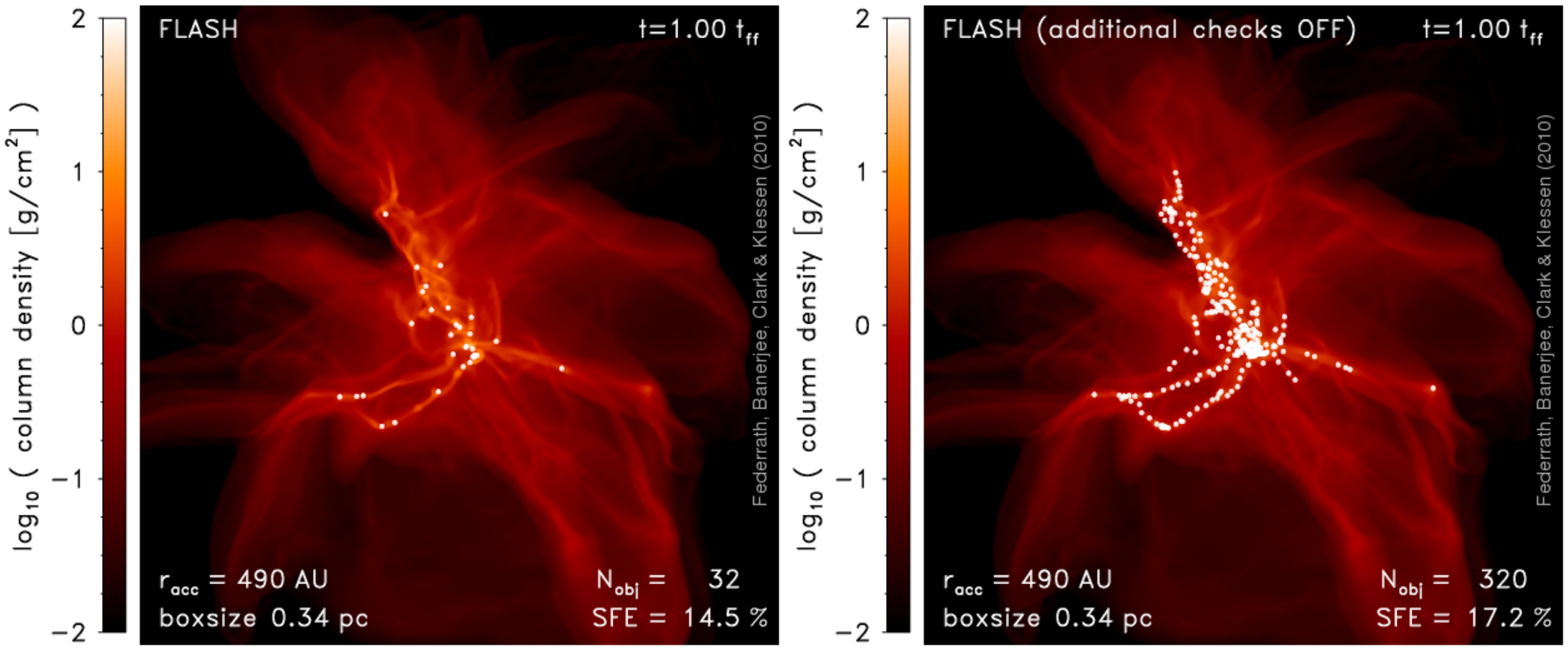


1. Cell exceeds density threshold, $\rho > \rho_{\text{res}}$
2. Highest level of AMR
3. Converging toward the center
4. Central minimum in gravitational potential
5. Jeans unstable, $|E_{\text{grav}}| > 2E_{\text{th}}$
6. Bound, $E_{\text{grav}} + E_{\text{th}} + E_{\text{kin}} + E_{\text{mag}} < 0$
7. Not within the accretion radius of an existing sink particle

Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Sink particle implementation in FLASH

Movies available: http://www.mso.anu.edu.au/~chfeder/pubs/sinks/sinks.html



all checks ON

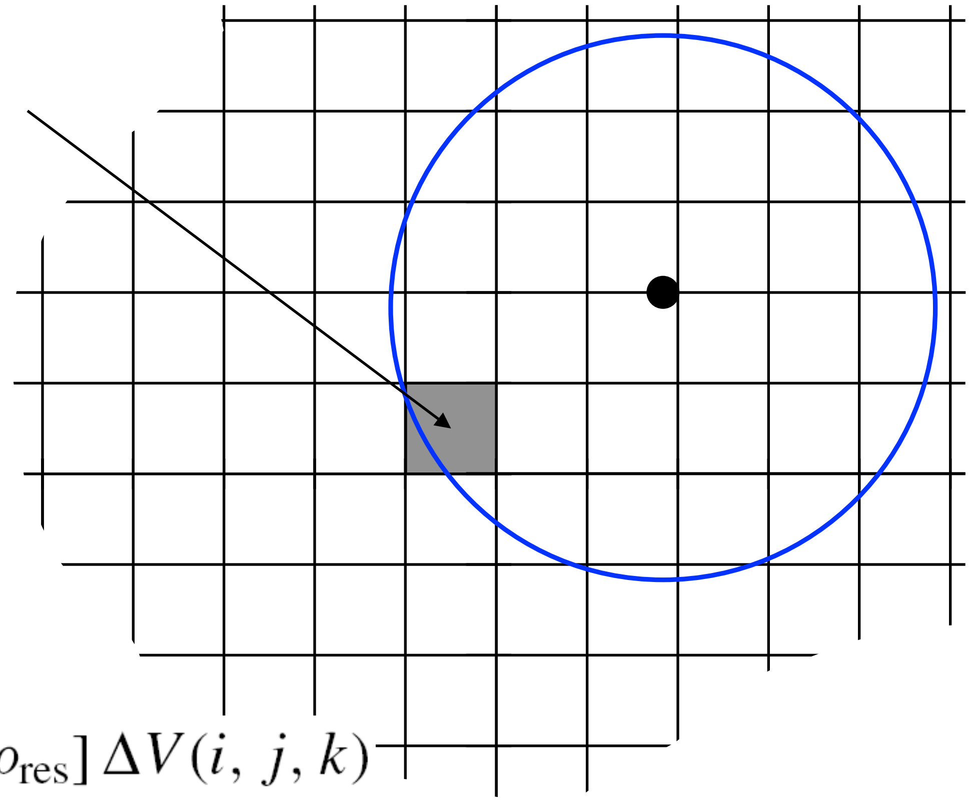
$\rho > \rho_{\mathrm{res}}$

only

Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Gas accretion

$$\rho > \rho_{\mathrm{res}}$$



1. Gas must be bound to the sink

2. Gas must be moving towards the sink

$$\Delta M = [\rho(i,j,k) - \rho_{\mathrm{res}}]\,\Delta V(i,j,k)$$

Mass, momentum, angular momentum conservation

Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Sink particle implementation in FLASH

Gas accretion

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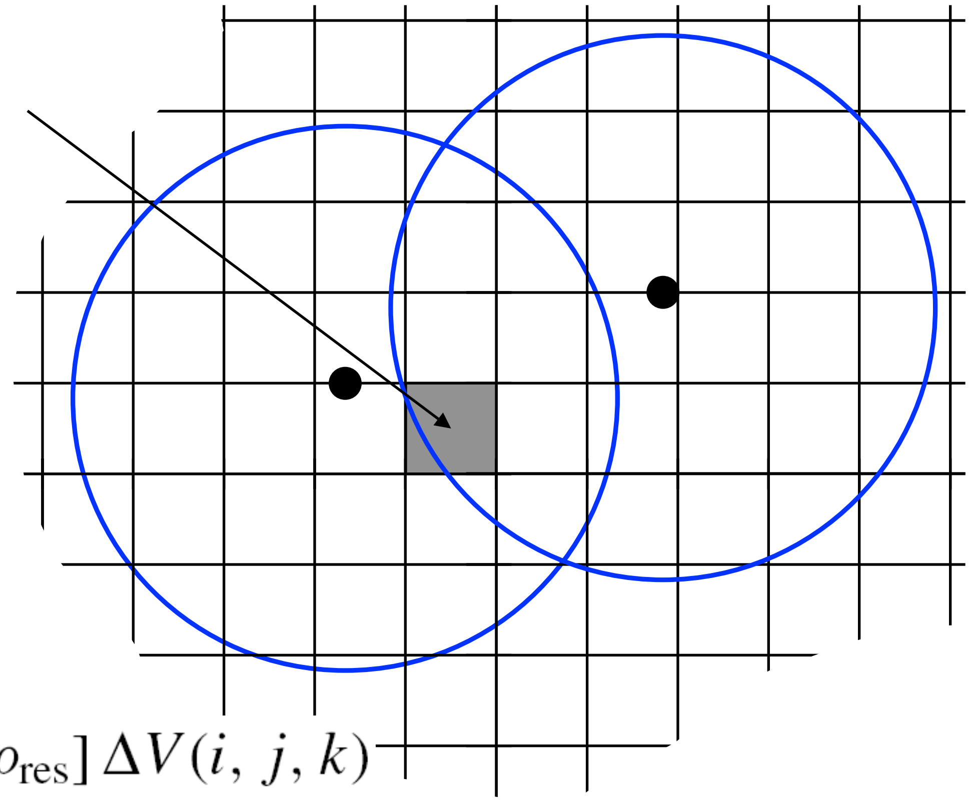


$$\Delta M = [\rho(i,j,k) - \rho_{\rm res}]\,\Delta V(i,j,k)$$

Mass, momentum, angular momentum conservation

Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Sink particle implementation in FLASH

Gravitational interactions

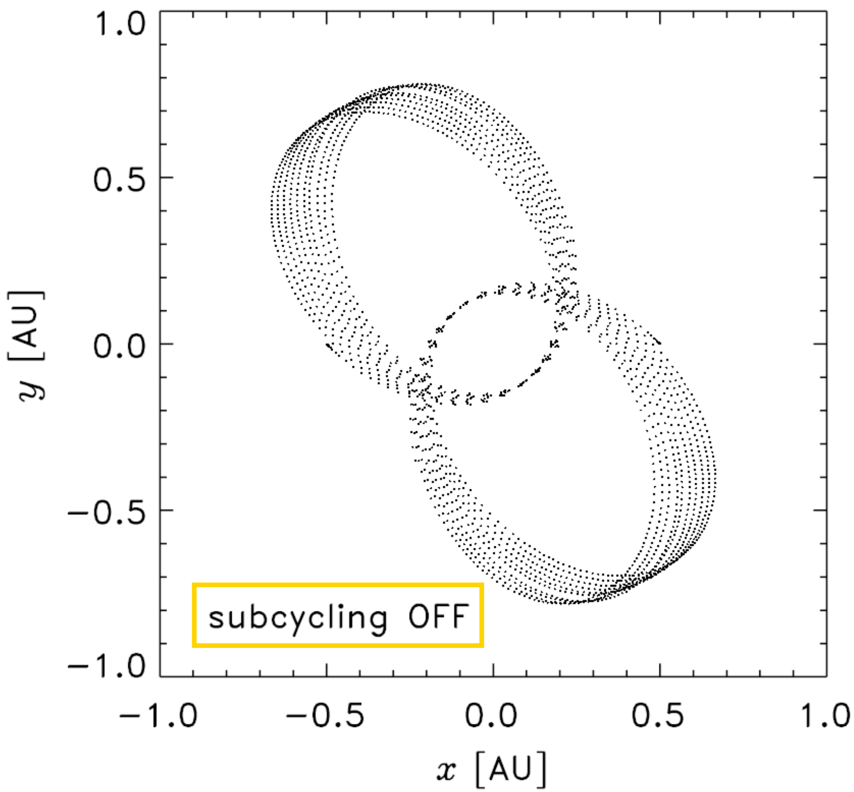
Gas—Gas	(multigrid solver, tree solver)
Gas—Sinks	(interpolation from grid)
Sinks—Gas	(direct summation, all cells)
Sinks—Sinks	(direct N-Body summation)

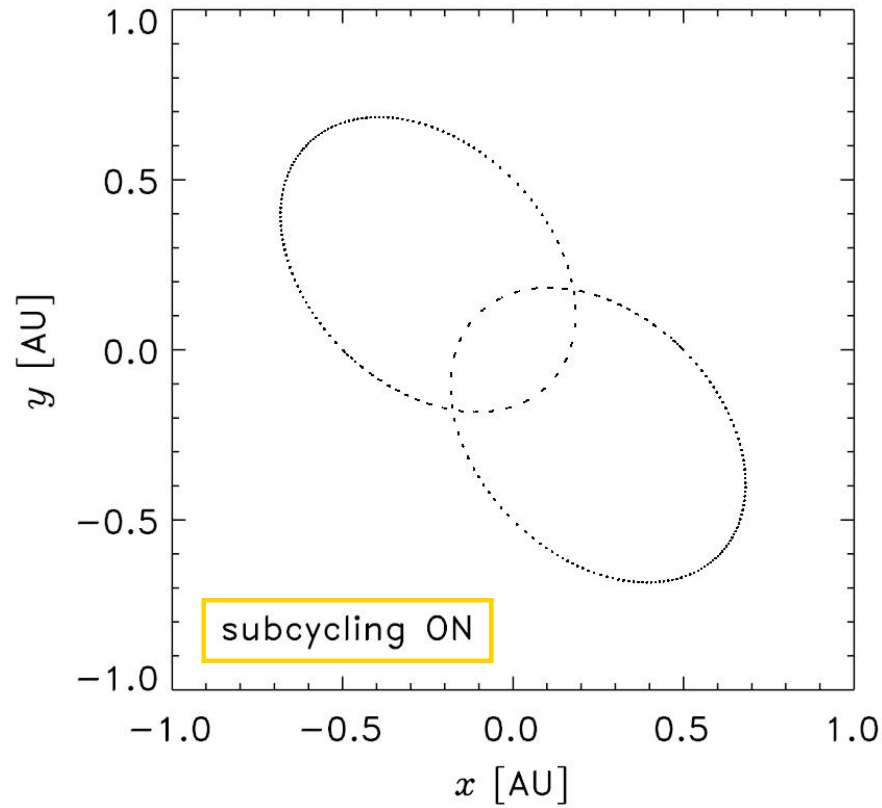
→ Strong constraints on timestep

→ Subcycling with Leapfrog required

Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

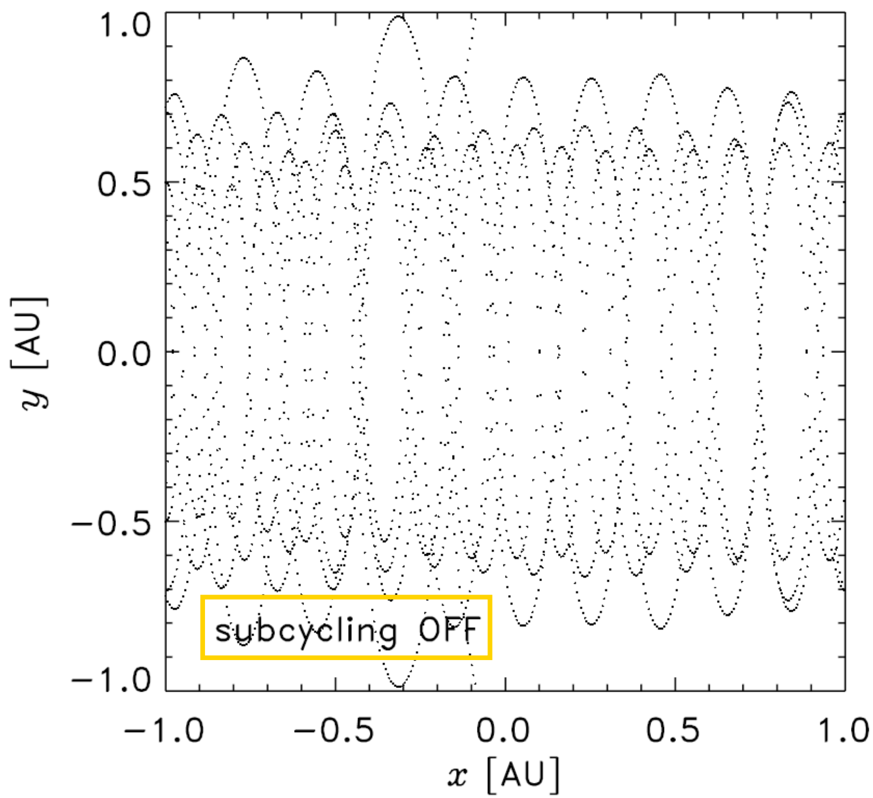
Subcycling required to capture N-Body dynamics

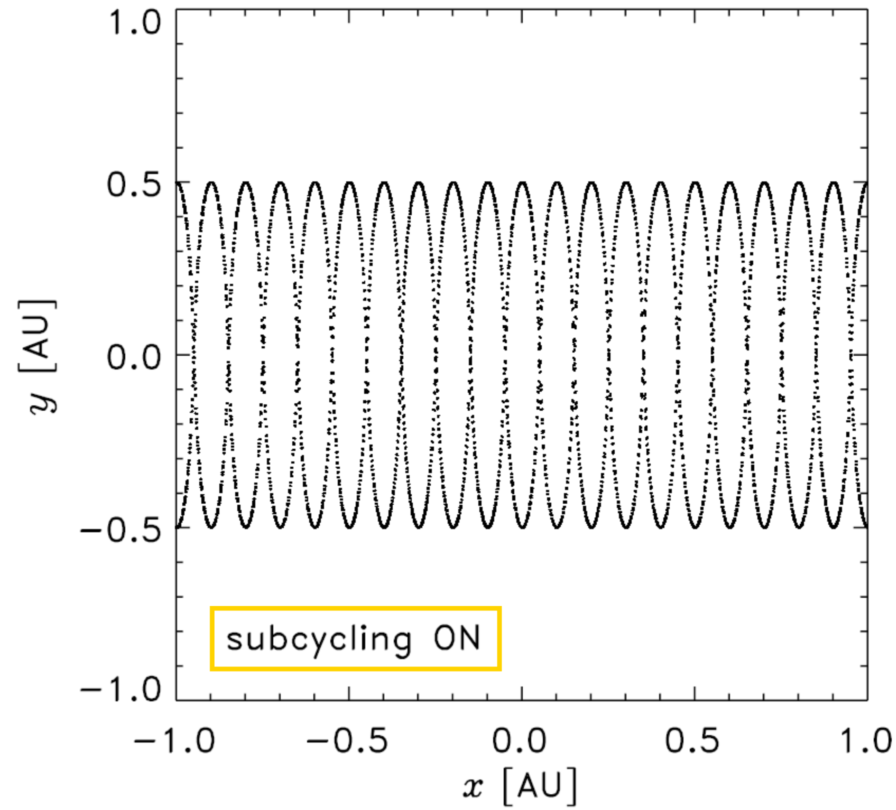




Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

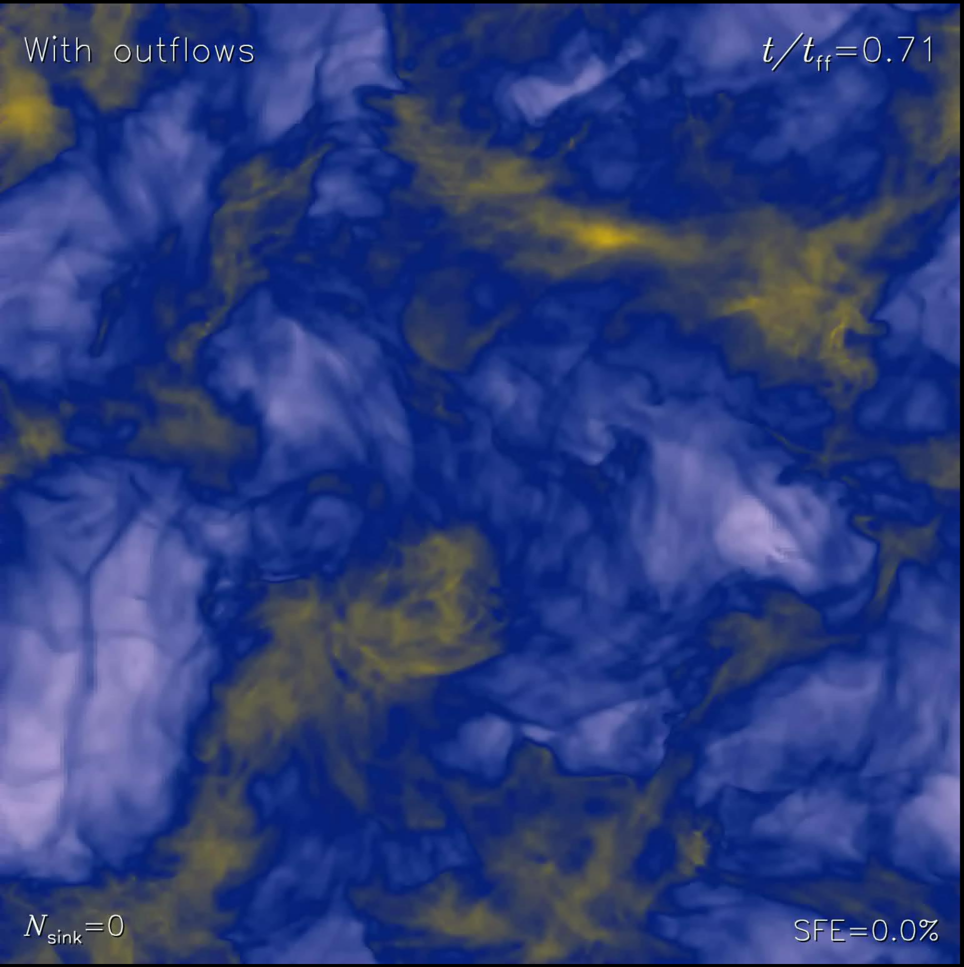
Subcycling required to capture N-Body dynamics





Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Grid-based Magnetohydrodynamics with Sink Particles

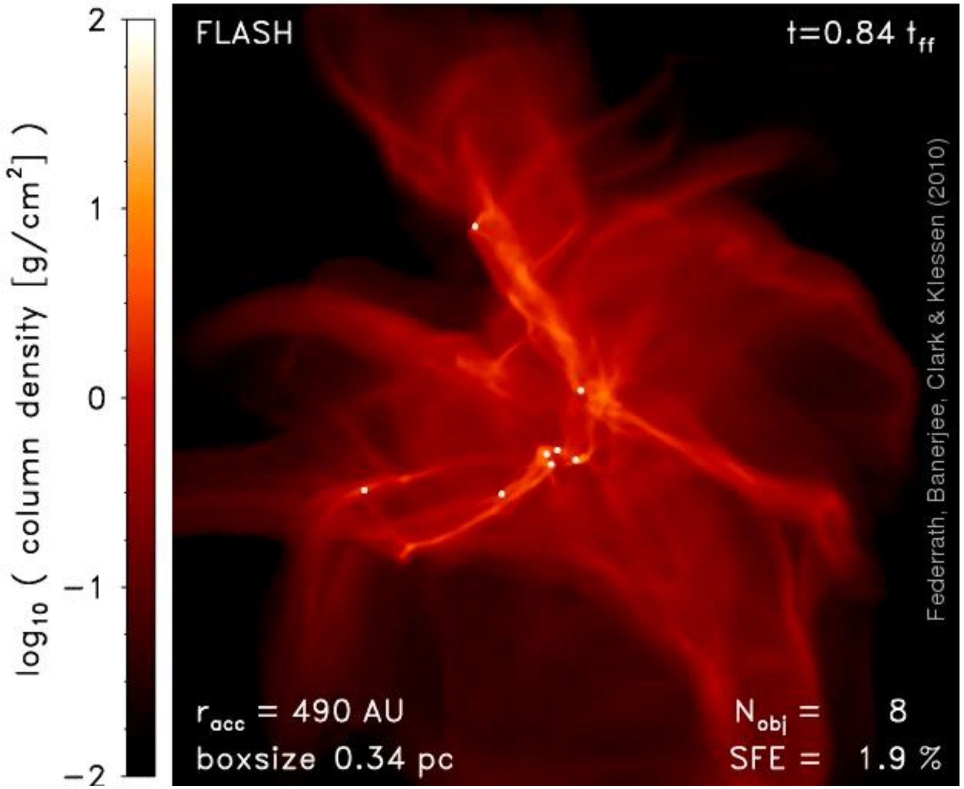


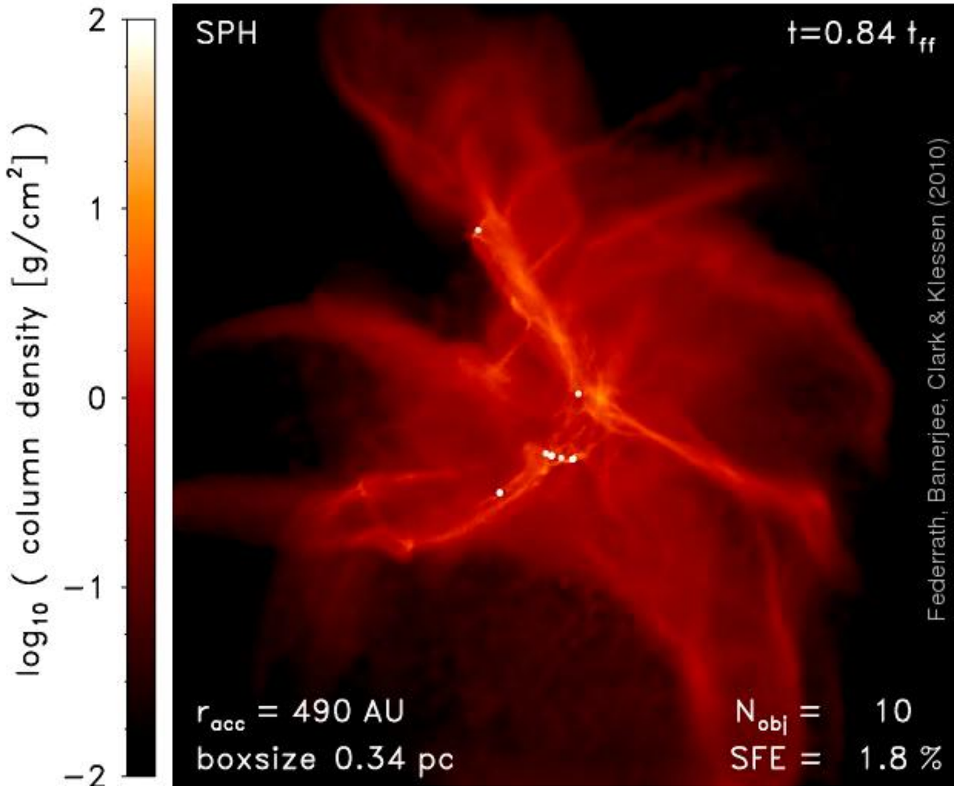
Movies available: http://www.mso.anu.edu.au/~chfeder/pubs/outflow_model/outflow_model.html

Sink particles: AMR versus SPH

Sink particles: AMR versus SPH

Movies available: http://www.mso.anu.edu.au/~chfeder/pubs/sinks/sinks.html





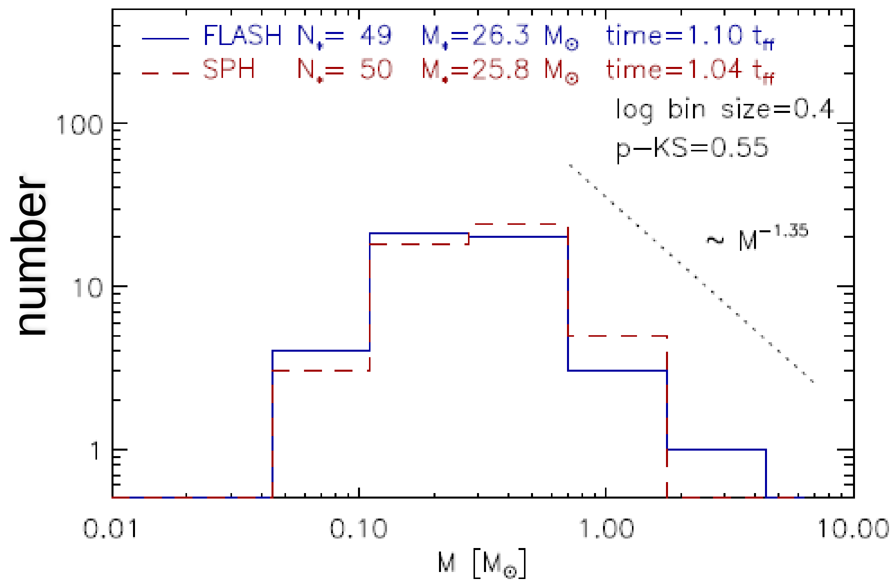
FLASH (AMR)

SPH

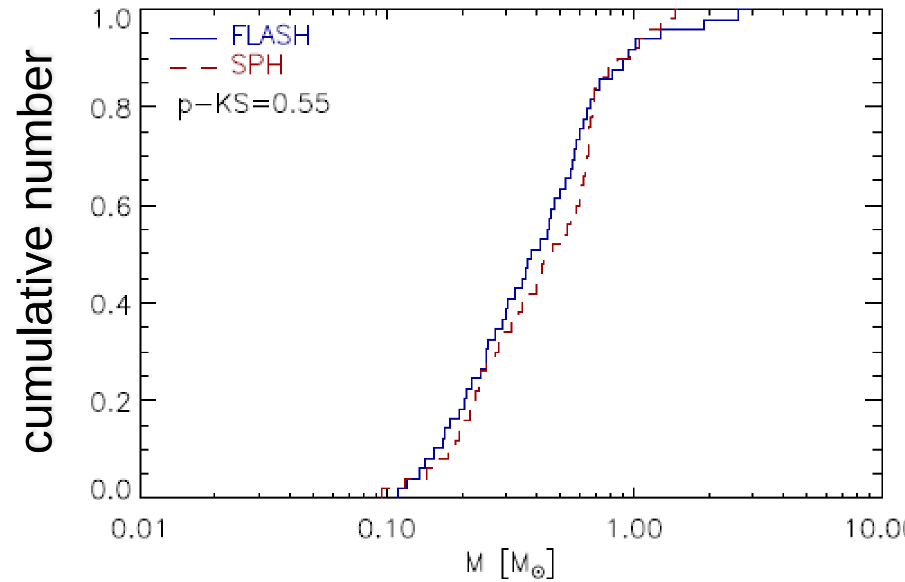
Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Comparison for SFE~26%

1. Sink mass functions agree well
2. Number of sinks: FLASH 49, SPH 50



sink mass



sink mass

Federrath, Banerjee, Clark, Klessen (2010, ApJ 713, 269)

Sink particle conclusions

(Federrath, Banerjee, Clark, Klessen 2010, ApJ 713, 269)

- Sink creation checks important to avoid spurious sinks in both SPH and AMR
- Encouraging agreement between FLASH and SPH-NG
- computational cost:

  FLASH: 10,300 CPU hours, run on 128 CPUs

  SPH-NG: 2,400 CPU hours, run on 16 CPUs

  (AMR: factor of 30 more resolution elements necessary in FLASH)

→ SPH is faster in collapse calculations

...but what about magnetic fields?

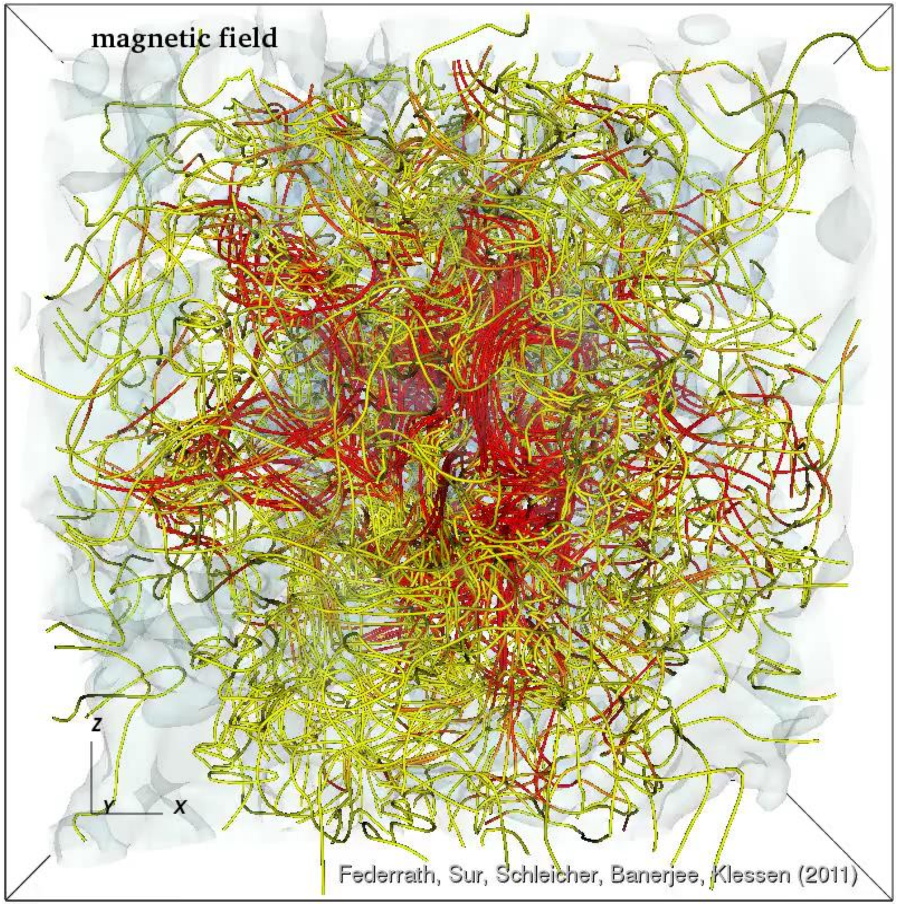
Magnetic fields in SPH and grid

- Problems with magnetic fields in SPH (Price & Federrath 2010)

- Divergence cleaning in SPMHD (Tricco & Price 2012)

http://www.mso.anu.edu.au/~chfeder/pubs/dynamo_grav/dynamo_grav.html

(Federrath, Sur, Schleicher, Banerjee, Klessen 2011, ApJ 731, 62)



Dynamo action

Strength and Weaknesses of SPH and grid

SPH

+ Automatic refinement on density
+ Typically faster in collapse calculations
+ More robust
+ Intrinsic mass conservation

- More complex data structure
- Problems with magnetic fields

Grid (AMR)

+ Simpler data structure (indexing)
+ Typically faster for pure hydro
+ Refinement on arbitrary quantities (e.g., position, shocks, etc.)
+ Magnetic fields

- Needs more resolution elements for collapse calculations (AMR)
- Sometimes less robust (solver crashes)

Unstructured Grid (e.g. AREPO)

Springel 2010