Grid-based hydrodynamics

Discretisation

$$x \rightarrow x_i \in \{x_1, x_2, \dots x_{N_x}\}$$

$$t \rightarrow t_n \in \{t_1, t_2, \dots t_{N_t}\}$$

- Denote quantity q_i^n as the value of q at time n and grid point i.

$$q(x_i, t_n) \equiv q_i^n$$

- In general, we want to know q_i^{n+1} from q_i^n
  - many possible ways to do that
  - many are unstable, and the stable ones are always somewhat diffusive $\Rightarrow$ try and find least dissipative, but stable method ($\rightarrow$ applied math)

Basic integration schemes

$$\underbrace{\partial_t}_{\text{time-deriv.}} q(x,t) + \overbrace{\partial_x}^{\text{space-deriv.}} \left[q(x,t) \cdot u(x,t) \right] = 0$$

could be ρ, $\rho\vec{u}$, ρe_{tot}

velocity

- **Time derivative**

Forward Euler method (simplest):

$$\partial_t q \longrightarrow \frac{q^{n+1} - q^n}{\Delta t} \quad \text{with } \Delta t = t^{n+1} - t^n$$

- Think of simple ODE: $\dot{q} = \partial_t q = f(q,t)$

$$\Rightarrow \frac{q^{n+1} - q^n}{\Delta t} = f(q^n, t^n)$$

$$\Rightarrow \boxed{q^{n+1} = q^n + \Delta t \cdot f(q^n, t^n)}$$

(explicit method) (1st-order method)

- becomes unstable if Δt is chosen too big
- could be made higher-order (Runge-Kutta, Adams)

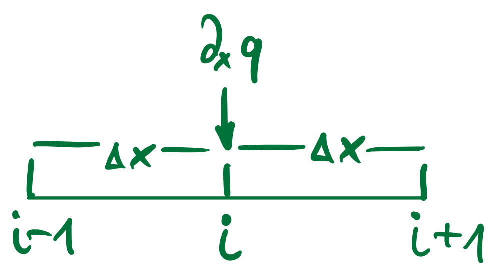
- **Spatial derivative**

$$\partial_x q = \lim_{\Delta x \to 0} \frac{q(x+\Delta x) - q(x)}{\Delta x}$$

- in discretised form:

$$\partial_x q \rightarrow \frac{q_{i+1} - q_{i-1}}{2\Delta x}$$

(2nd order accurate)



(Centred Difference)

Advection

- Start with simpler form of hydro eq., where $u = \text{const.}$

$$\Rightarrow \quad \partial_t q + u \, \partial_x q = 0$$

- Insert time- and space-derivatives:

$$\frac{q_i^{n+1} - q_i^n}{\Delta t} + u \, \frac{q_{i+1}^n - q_{i-1}^n}{2 \Delta x} = 0$$

$$\Rightarrow \quad \boxed{q_i^{n+1} = q_i^n - \frac{\Delta t}{2 \Delta x} u \left(q_{i+1}^n - q_{i-1}^n \right)}$$

(Centred Difference Scheme)

unstable !

Instead **upwind scheme**:

$$\boxed{q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x} u \left(q_{i-1}^n - q_i^n \right)} \quad (u > 0)$$

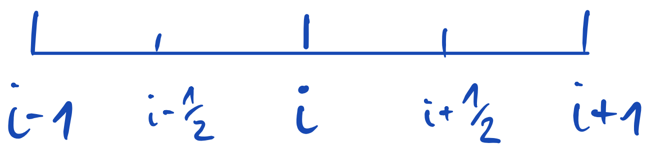
(Upwind Difference Scheme) $\Rightarrow$ stable

- upwind scheme stable, but diffusive $\rightarrow$ numerical diffusion.

- Diffusion eq.: $\boxed{\frac{\partial}{\partial t} q} - D \boxed{\frac{\partial^2}{\partial x^2} q} = 0$

discretised version:

$$\boxed{\frac{q_i^{n+1} - q_i^n}{\Delta t}} - D \boxed{\frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{\Delta x^2}} = 0$$



- Now take ∂_x part of the upwind scheme:

$$\frac{q_i - q_{i-1}}{\Delta x} = \underbrace{\boxed{\frac{q_{i+1} - q_{i-1}}{2\Delta x}}}_{\text{centred diff.}} - \Delta x \underbrace{\boxed{\frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{2\Delta x^2}}}_{\text{diffusion}}$$

centred diff.

diffusion with a diffusion constant $D = \frac{\Delta x \cdot u}{2}$

$\Rightarrow$ upwind scheme is the centred difference scheme with diffusion.

CFL condition (time step criterion)

- explicit scheme is unstable if Δt chosen too big.
  → because information would flow across more than 1 cell width Δx in 1 Δt.

$$\Rightarrow \boxed{\Delta t < \underbrace{CFL}_{[0,1]\ \text{(usually pick 0.8)}} \cdot \frac{\Delta x}{\max\left[\mathrm{abs}(u) + c_s\right]}}$$

(CFL condition)