Viscosity, Reynolds Number, Turbulence

- Start from the momentum eq. (in index notation):

$$\partial_t (\rho v_i) + \partial_i \Pi_{ij} = 0 \qquad (1)$$

with $\Pi_{ij} = \underbrace{\rho v_i v_j}_{= \rho(\vec{v} \otimes \vec{v})} + \delta_{ij} P$

- So far we have ignored viscosity, but real gases/fluids have viscosity, which can dissipate kinetic energy.
  $\Rightarrow$ Add viscous processes via the rate-of-strain tensor σ_{ij}:

$$\Pi_{ij} \rightarrow \Pi_{ij} - \sigma_{ij} \, ,$$

where $\sigma_{ij} = \mu \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \underbrace{\partial_k v_k}_{= \mathrm{div}\, \vec{v}} \right)$

(traceless rate-of-strain tensor)

(μ: dynamic viscosity)

$$[\mu] = \frac{\text{mass}}{\text{length} \times \text{time}}$$

(expansion/compression via $\mathrm{div}\,\vec{v}$ subtracted, so what remains are shearing motions)

- Insert this into Eq. (1):

$$\partial_t v_i + \underbrace{v_j \partial_j v_i}_{= (\vec{v}\cdot\nabla)\vec{v}} = -\frac{1}{\rho}\partial_i P + \frac{1}{\rho}\partial_i \sigma_{ij}$$

- If we further assume that $\mu = \text{const.}$, we get

$$\boxed{\partial_t \vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{v} + \frac{\mu}{3\rho}\left[\nabla(\nabla\cdot\vec{v})\right]}$$

(Navier-Stokes equation)

or $\partial_t(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) = -\nabla P + \nabla\cdot(2\nu\rho\mathbb{S})$

with $S_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i) - \frac{1}{3}\delta_{ij}(\nabla\cdot\vec{v})$

and the kinematic viscosity $\boxed{\nu = \mu/\rho}$ (units cm^2/s).

- Now we use the Navier-Stokes eq. to define the Reynolds number: $Re = \dfrac{V\cdot L}{\nu}$

- Think of Re as the ratio of the non-linear term and the dissipative term in the Navier-Stokes eq.:

$$\left.\begin{aligned} (\vec{v}\cdot\nabla)\vec{v} &\rightarrow \sim \frac{V^2}{L} \\ \nu\nabla^2\vec{v} &\rightarrow \sim \frac{\nu V}{L^2} \end{aligned}\right\} \quad \text{ratio: } Re = \frac{VL}{\nu}$$

- If $Re \gg 1 \Rightarrow$ Turbulent Flow, otherwise Laminar Flow.
- Transition from laminar to turbulent at $\approx Re \approx 1000$, and for $Re \approx 100$: Karman-vortex street.

Kolmogorov spectrum

- $\boxed{E(k) \sim k^{-5/3}}$ Where from?

- kinetic energy per unit mass (Kolmogorov turbulence is incompressible, so $\rho = \text{const}$):

$$E \sim v^2$$

- Energy dissipation rate is constant through the cascade (change of E with time):

$$\dot{E} \sim \frac{v^2}{t} \quad \text{and} \quad t \sim \frac{L}{v} \quad \text{(eddy turnover time)}$$

Assumption: $\boxed{\dot{E} = \text{const.}}$ (Kolmogorov 1941)

$$\Rightarrow \dot{E} \sim \frac{v^3}{L} \overset{!}{=} \text{const.} \Rightarrow \boxed{v \sim L^{1/3}}$$

- Then the energy spectrum is

$$E(k) \sim \frac{dv^2}{dk} \sim \frac{v^2}{k} \sim \frac{L^{2/3}}{k} \sim \frac{k^{-2/3}}{k} \sim k^{-5/3}.$$