Magnetohydrodynamics (MHD)

- First we assume that the fluid is almost completely neutral, with only a small charge density ρ_e present. This condition can be written

$$\boxed{\rho_e v \sim \frac{v^2}{c^2} J, \qquad (1)}$$

where ρ_e is small. (Will check Eq. (1) at the end.)
Here, $\vec{v}$ is the gas velocity, and J is the current density, which has units of current per unit area.

- Assumption Eq. (1) is reasonable, because for a good conductor, any net charge would immediately be neutralised by a current. An important consequence of this assumption is that in MHD there are practically no electric fields, i.e., any charge separation would lead to currents that immediately destroy the $\vec{E}$ field. However, the currents produce a magnetic field $\vec{B}$; hence the name

"Magnetohydrodynamics".

- Now we need to couple Maxwell's equations (in the non-relativistic approximation, i.e., $v^2/c^2 \approx 0$) with the momentum eq. of hydrodynamics.

- Maxwell's equations (in CGS units):

$$\boxed{\begin{aligned}
\nabla\cdot\vec{E} &= 4\pi\rho_e && (2)\\
\nabla\cdot\vec{B} &= 0 && (3)\\
\nabla\times\vec{E} &= -\frac{1}{c}\,\partial_t\vec{B} && (4)\\
\nabla\times\vec{B} &= \frac{4\pi}{c}\,\vec{j} + \frac{1}{c}\,\partial_t\vec{E} && (5)
\end{aligned}}$$

- In a highly conducting fluid, we expect the fields to vary on the same length L and timescale T as the fluid itself, where $L/T \sim v$ is the fluid velocity, i.e., the $\vec{B}$ field is basically produced/maintained by the fluid motion.

- Approximating $\nabla \to L^{-1}$ and $\partial_t \to T^{-1}$, the 3rd Maxwell eq., $\nabla\times\vec{E} = -1/c\,\partial_t\vec{B}$, says that the $\vec{E}$ field is a factor v/c smaller than the $\vec{B}$ field:

$$\boxed{E \sim \frac{v}{c}\,B \qquad (6)}$$

- Using this, we see that the „displacement current" in the last Maxwell eq. (the term $1/c\,\partial_t\vec{E}$) can be neglected, because it is of order v^2/c^2:

$$\Rightarrow \boxed{\nabla\times\vec{B} = \frac{4\pi}{c}\,\vec{j} \qquad (7)}$$

- Now we need to couple this to the fluid equations. First, the EM force is the Lorentz force (per unit volume):

$$\boxed{\vec{F}_L = \rho_e \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \qquad (8)}$$

- Thanks to our assumption, Eq. (1), and using Eq. (6), we see the $\vec{E}$ field term in Eq. (8) is small and can be neglected, hence

$$\boxed{\vec{F}_L = \frac{1}{c} \vec{j} \times \vec{B} \qquad (9)}$$

- Thus, the Euler eq. with the addition of the Lorentz force becomes

$$\boxed{\rho \frac{D\vec{v}}{Dt} = -\nabla P + \frac{1}{c} \vec{j} \times \vec{B} \qquad (10)}$$

- Using Eq. (7), $\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}$, this becomes

$$\rho \frac{D\vec{v}}{Dt} = -\nabla P + \frac{1}{4\pi} \left(\nabla \times \vec{B}\right) \times \vec{B}$$

$$\Longleftrightarrow \rho \frac{D\vec{v}}{Dt} = -\nabla P + \frac{1}{4\pi} \left[-\frac{1}{2} \nabla B^2 + \left(\vec{B} \cdot \nabla\right) \vec{B} \right]$$

$$\Longleftrightarrow \boxed{\rho \frac{D\vec{v}}{Dt} = -\nabla \left(P + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} \left(\vec{B} \cdot \nabla\right) \vec{B} \qquad (11)}$$

(MHD Euler Equation)

- The term $B^2/8\pi$ is the magnetic pressure, while the $1/4\pi\,(\vec{B}\cdot\nabla)\vec{B}$ term is the magnetic tension ($\rightarrow$ see later).
- Now we know how $\vec{B}$ affects the motion of the gas, but how do the gas motions affect/generate $\vec{B}$?
- We start from the non-relativistic Ohm's Law:

$$\boxed{\vec{j} = \sigma\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \qquad (12)}$$

σ is the conductivity

- Substituting Eq. (12) into Eq. (7) gives

$$\boxed{\vec{E} = \frac{c}{4\pi\sigma}\nabla\times\vec{B} - \frac{1}{c}\vec{v}\times\vec{B} \qquad (13)}$$

- Then this into the 3rd Maxwell eq.:

$$\boxed{\partial_t\vec{B} = -c\,\nabla\times\vec{E} = \nabla\times(\vec{v}\times\vec{B}) - \frac{c^2}{4\pi\sigma}\nabla\times(\nabla\times\vec{B}) \qquad (14)}$$

- We can expand the 2nd term on the RHS:

$$\partial_t\vec{B} = \nabla\times(\vec{v}\times\vec{B}) - \frac{c^2}{4\pi\sigma}\left[\nabla\left(\cancelto{0}{\nabla\cdot\vec{B}}\right) - \Delta\vec{B}\right]$$

(2nd Maxwell)

$$\Rightarrow \boxed{\partial_t\vec{B} = \nabla\times(\vec{v}\times\vec{B}) + \frac{c^2}{4\pi\sigma}\Delta\vec{B} \qquad (15)}$$

$= \eta$ (magnetic resistivity)

(MHD induction equation)

- We see in Eq. (15) that the $\Delta\vec{B}$ term with the magnetic resistivity η is a diffusion term ($\partial_t q - \partial_x^2 q = 0$), which leads to diffusion of $\vec{B}$ over time. $\Longrightarrow$ magnetic (Ohmic) diffusion.

- In the limit of infinite conductivity $\sigma \to \infty$, Eq. (15) simplifies to ($\eta \to 0$):

$$\boxed{\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}) \qquad (16)}$$

(Induction eq. in the limit of ideal MHD)

- Recall that even if the ionisation fraction is very small (say 10^{-8}, e.g., in molecular clouds), the conductivity can still be very high, such that the ideal MHD approximation is good for many astrophysical gases.

- However ideal MHD breaks down for example in the dense parts of accretion discs, where non-ideal MHD is active ($\to$ Ohm, Ambipolar, Hall diffusion).

- Note that $\vec{J}$ and $\vec{E}$ do not occur explicitely in the MHD equations, but are still there, and given by Eq. (12) and (13).

- Also note $\operatorname{div} \vec{B} = 0$ ($\to$ immediately clear from Eq. 14).

- Finally, let's check for consistency with our very first assumption (ρ_e small), Eq. (1): $\rho_e v \sim \frac{v^2}{c^2} j$.

- Using the 1st Maxwell eq.:

$$\rho_e = \frac{1}{4\pi} \nabla \cdot \vec{E} \overset{(\text{Eq.}13)}{=} \frac{1}{4\pi} \nabla \cdot \Big[\underbrace{\frac{c}{4\pi\sigma} \nabla \times \vec{B}}_{=0} - \frac{1}{c} \vec{v} \times \vec{B} \Big]$$

$$\Longrightarrow \rho_e \sim \nabla \cdot \left(\frac{1}{c} \vec{v} \times \vec{B} \right)$$

- Now again $\nabla \to L^{-1}$ $\Rightarrow$ $\rho_e \sim \frac{v}{c} \frac{1}{L} B$
- And from Eq. (7), $\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}$ $\Rightarrow$ $B \sim \frac{L}{c} j$ $\Bigg\} \Rightarrow$

$$\Longrightarrow \rho_e \sim \frac{v}{c} \frac{1}{c} j$$

or $\boxed{\rho_e v \sim \frac{v^2}{c^2} j}$, which is Eq. (1).

Finally, the complete set of MHD equations:

mass: $\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$

velocity: $\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla \left(P + B^2/8\pi \right) + \frac{1}{4\pi\rho} (\vec{B} \cdot \nabla) \vec{B}$

energy: $\partial_t (\rho e_{tot}) + \nabla \cdot \left[(\rho e_{tot} + P_{tot}) \vec{v} - \frac{1}{4\pi} (\vec{B} \cdot \vec{v}) \vec{B} \right] = 0$

induction: $\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma} \Delta \vec{B}$; div-free: $\nabla \cdot \vec{B} = 0$

with $\rho e_{tot} = \frac{1}{2} \rho v^2 + \rho e_{int} + B^2/8\pi$ and $P_{tot} = P + B^2/8\pi$