Sedov explosion (similarity solution)

- The external radius of the explosion is given by

$$\boxed{r = \left(\frac{Et^2}{\rho_1}\right)^{1/5}}$$

[More generally, if the ambient density is $\rho(r) \sim r^{-\omega}$, then

$$r = \left(\frac{Et^2}{\rho_1}\right)^{\frac{1}{D+2-\omega}}, \text{ with } D = 1,2,3 \text{ for 1D, 2D, 3D}$$

(Book 1994)],

which suggests that we can introduce a dimensionless radius

$$\xi = \frac{r}{R} \qquad \text{(R is the shock position)}$$

and use ξ to express $v(r,t)$, $\rho(r,t)$, and $P(r,t)$.

$$\Rightarrow v(\xi,t) = \frac{2}{5}\frac{r}{t} V(\xi), \quad \text{(normalised velocity profile)}$$

where $V(1) = \dfrac{2}{\gamma+1}$, because $v_s = 1$ @ $\xi = 1$, i.e., where $r = R$ (inner edge of shock)

$\left(\text{from } u = \dfrac{2v_s}{\gamma+1}\right)$

- Similarly: $\rho = \rho_1 G(\xi)$ with $G(1) = \dfrac{\gamma+1}{\gamma-1}$

(normalised density profile) $\left(\text{from } \dfrac{\rho_1}{\rho_2} = \dfrac{\gamma-1}{\gamma+1}\right)$, and $$c_s^2 = \frac{\gamma P_2}{\rho_2} \underset{\substack{(1)\\ \text{(from prev. lecture)}}}{=} \frac{2\gamma\,\rho_1}{(\gamma+1)\,\rho_2} V_s^2 \underset{\left(V_s = \frac{2}{5}\frac{r}{t}\right)}{=} \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\left(\frac{2}{5}\frac{r}{t}\right)^2$$

or $$c_s^2 = \frac{4}{25}\frac{r^2}{t^2}\, Z(\xi)$$ (normalised sound speed^2 profile)

with $$Z(1) = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}.$$

- Now we use energy conservation to derive the functions $V(\xi)$, $G(\xi)$, and $Z(\xi)$.

- Per unit time dt, a sphere with radius r changes its energy according to the energy eq., $\partial_t(\rho e_{tot}) + \nabla\cdot\left[(\rho e_{tot}+P)\vec{v}\right] = 0$

flux through surface $4\pi r^2 \times dt$

$$\underset{\text{(total energy change)}}{dE} = \underset{\text{(integrated over surface)}}{\boxed{4\pi r^2 \rho v\left(\frac{v^2}{2}+h\right) dt}}$$

with the enthalpy $$h = \frac{P}{\rho}\,\frac{\gamma}{\gamma-1}$$

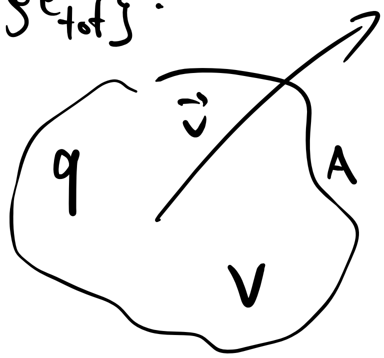
- The amount of energy gained by expansion of the shell is

$$\boxed{\underbrace{4\pi r^2 \underbrace{V_s\, dt}_{dr}}_{d(\text{volume})}\left(e_{int} + \frac{v^2}{2}\right)\rho}$$

Comment on energy conservation eq. in expanding shell

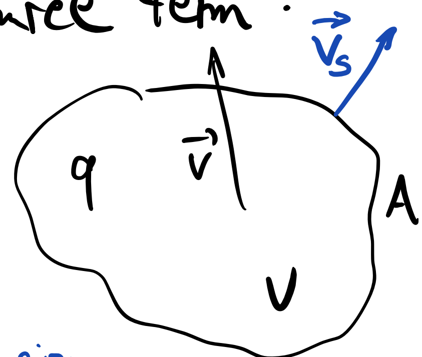
When we derived the hydro eqs., we considered a fixed volume (V). Fluxes across the surface (A) in or out of the volume determine the change (in time) of any conserved quantity q, e.g., $q = \{\rho, \rho\vec{v}, \rho e_{tot}\}$:

$$\frac{\partial}{\partial t}\int_V q\, dV + \int_A q\vec{v}\, d\vec{A} = 0.$$



Now we have an expanding volume for which the surface expands with a speed v_s. This requires us to treat the swept-up material as a source term:

$$\underbrace{\frac{\partial}{\partial t}\int_V q\, dV + \int_A q\vec{v}\, d\vec{A}}_{\text{original conservation law}} = \underbrace{\int_A q\vec{v}_s\, d\vec{A}}_{\text{Source term due to expansion}}$$



- For example, take the energy equation: $q = \rho e_{tot} = \rho\left[\frac{1}{2}v^2 + e_{int}\right]$

(and also consider additional source term because of pressure
$\Rightarrow$ flux term is modified to $\rho e_{tot} + P = \rho\left[\frac{1}{2}v^2 + h\right]$)

$$\Rightarrow \cancelto{0}{\frac{\partial}{\partial t}\int_0^R \rho e_{tot}\, dV} + \int_\Omega \left[\rho v\left(\frac{1}{2}v^2 + h\right) - v_s \rho e_{tot}\right] r^2 d\Omega = 0$$

here: $\rho e_{tot} = \text{const}$ (energy-conserving phase)

• Using the energy conservation in an expanding shell, we thus find (or equating the energy change per dt with the gain)

$$\boxed{v\left(\frac{v^2}{2}+h\right) = \frac{2}{5}\,\frac{r}{t}\left(\frac{v^2}{2}+e_{int}\right)} \quad \cancel{(\times\rho)}$$

$$h = \frac{p}{\rho}\,\frac{\gamma}{\gamma-1} = \frac{c_s^2}{\gamma-1} \qquad\qquad \frac{p}{\rho}\,\frac{1}{\gamma-1} = \frac{c_s^2}{\gamma(\gamma-1)}$$

$$\Rightarrow v\left(\frac{v^2}{2}+\frac{c_s^2}{\gamma-1}\right) = \frac{2}{5}\,\frac{r}{t}\left(\frac{v^2}{2}+\frac{c_s^2}{\gamma(\gamma-1)}\right)$$

Now insert $v = \frac{2}{5}\frac{r}{t}V(\xi)$ and $c_s^2 = \left(\frac{2}{5}\frac{r}{t}\right)^2 Z(\xi)$:

$$V\left(\frac{V^2}{2}+\frac{Z}{\gamma-1}\right) = \frac{V^2}{2}+\frac{Z}{\gamma(\gamma-1)}$$

$$\Rightarrow \boxed{Z = \frac{\gamma(\gamma-1)(1-V)\,V^2}{2(\gamma V-1)} \qquad (1)}$$

(from energy equation) (relation between V and Z)

• Now we also need to do this for $G(\xi)$, using mass and momentum conservation in expanding shell, which gives the additional relations:

$$\boxed{\xi V' - (1-V)\frac{\xi G'}{G} + 3V = 0 \qquad (2)}$$

(primed quantities denote derivatives w.r.t. ξ) (from mass conservation) and $$\boxed{\frac{\xi Z'}{Z} + (1-\gamma)\frac{\xi G'}{G} + \frac{5-2V}{1-V} = 0} \quad (3)$$

(from momentum conservation)

- Finally, combining (1), (2), and (3) yields:

$$\boxed{\frac{\xi V'}{V} = \frac{\gamma(1-3\gamma)V^2 + (8\gamma-1)V - 5}{\gamma(\gamma+1)V^2 - 2(\gamma+1)V + 2}},$$

which has an analytic solution, found by Sedov in 1941 (see also von Neumann 1947 and Taylor 1950; details in Landau + Lifschitz: "Hydrodynamics", § 106).

Sedov solution:

$$\xi^5 = \left[\frac{\gamma+1}{2}V\right]^2 \left\{\frac{\gamma+1}{\gamma-1}\left[5-(3\gamma-1)V\right]\right\}^{\nu_1} \left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^{\nu_2}$$

$$G = \frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^{\nu_3} \left\{\ldots\right\}^{\nu_4} \left[\ldots\right]^{\nu_5}$$

with $\nu_1 = -\frac{13\gamma^2 - 7\gamma + 12}{(3\gamma-1)(2\gamma+1)}$; $\nu_2 = \frac{5\gamma - 5}{2\gamma+1}$;

$\nu_3 = \frac{3}{2\gamma+1}$; $\nu_4 = -\frac{\nu_1}{2-\gamma}$; $\nu_5 = -\frac{2}{2-\gamma}$