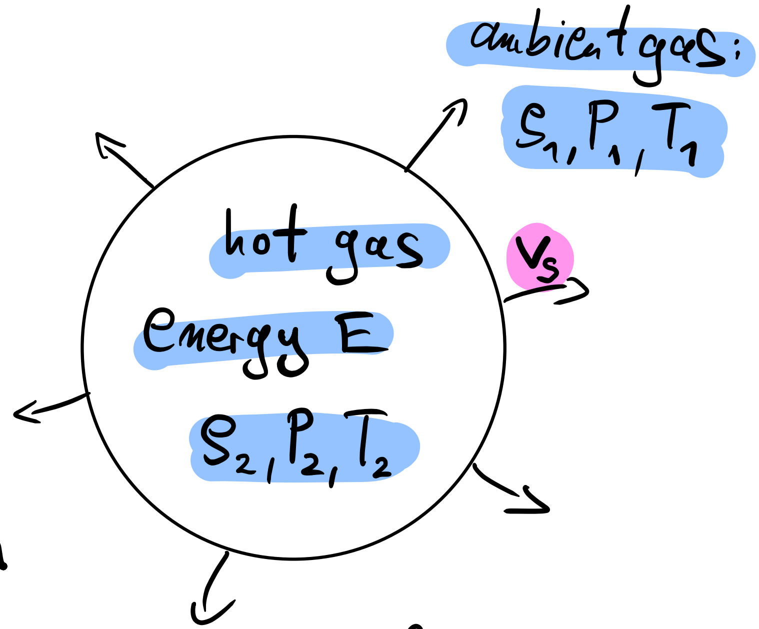
The Sedov Solution

Dimensional Analysis



- hot bubble with energy E expands into a homogeneous medium with density ρ_1

- Example: Supernova explosion
  - → in a very short time, an energy E is released in the centre
  - → we assume strong shock conditions ($u \gg c_s$) and we neglect the ambient pressure ($P_1 \ll P_2$); we also use a polytropic EOS with γ

- Under these conditions, the shock jump condition for the density is

$$\frac{\rho_1}{\rho_2} = \frac{P_1(\gamma+1) + P_2(\gamma-1)}{P_1(\gamma-1) + P_2(\gamma+1)} \overset{(P_1 \ll P_2)}{\approx} \frac{\gamma-1}{\gamma+1},$$

which implies that ρ_1 and ρ_2 are completely determined by one another. And since $\rho_1 = \text{const}$, so is ρ_2.

- Now consider a similarity solution that can be constructed from the dimensional variables E and ρ_1, and that evolves in time (t). We are searching for a dimensionless radius ($\tilde{r}$), with the following ansatz:

$$\tilde{r} = r \cdot t^{l} \cdot \rho_1^{m} \cdot E^{n} \qquad (0)$$

- Dimensional analysis yields:

$$\tilde{r}^{0} = L^{1} \cdot T^{l} \cdot M^{m} L^{-3m} \cdot M^{n} L^{2n} T^{-2n},$$

with length unit L, time unit T, and mass unit M.

- Equivalently:

$$\left.\begin{array}{lll} \text{for } L: & 0 = 1 - 3m + 2n \\ \text{for } T: & 0 = l - 2n \\ \text{for } M: & 0 = m + n \end{array}\right\} \Rightarrow$$

$$\Rightarrow l = -\frac{2}{5} \; ; \quad m = \frac{1}{5} \; ; \quad n = -\frac{1}{5}$$

- The only quantity with the dimension of a length that can be formed from E, t, and ρ_1 is

$$\left(\frac{E t^2}{\rho_1}\right)^{1/5} = [L] \,, \qquad \left(\text{From Eq.(0)} \Rightarrow r \sim E^{1/5} t^{2/5} \rho_1^{-1/5} \right)$$

which makes us set

$$r(t) = \tilde{r}_0 \left(\frac{E t^2}{\rho_1}\right)^{1/5} ,$$

with a dimensionless constant $\tilde{r}_0$.

- The shock velocity is

$$v_s = \frac{dr}{dt} = \tilde{r}_0 \left(\frac{E}{\rho_1}\right)^{1/5} \frac{2}{5} t^{2/5 - 1} = \frac{2}{5} \frac{r}{t} .$$

- We now use the shock velocity as a function of the piston speed u derived earlier:

$$u = \frac{2 v_s}{\gamma + 1} ,$$

from which the pressure within the shock ('behind' the shock $\hat{=}$ in the post-shock medium):

$$P_2 = P_1 \frac{\gamma(\gamma+1) u^2}{2 c_s^2} = P_1 \frac{2\gamma v_s^2}{(\gamma+1)\, c_s^2} = \frac{2 \rho_1 v_s^2}{\gamma + 1} \qquad (1)$$

$$\frac{\gamma P_1}{\rho_1}$$ sound speed^2 in ambient medium

- Also from the earlier expression for the shock velocity:

$$\Rightarrow V_s = \frac{2}{5}\frac{r}{t} \sim t^{2/5 - 1} \sim t^{-3/5}$$

$$\Rightarrow P_2 \sim V_s^2 \sim t^{-6/5}$$

$$\Rightarrow u \sim V_s \sim t^{-3/5}$$

- We can interpret these relations as follows:

A shock wave driven by the release of energy E propagates outwards with the t-dependent radius $r(t)$ and collects material with mass

$$M = \varrho_1 r^3 .$$

This material is accelerated from $v=0$ to $v \sim \frac{r}{t}$, such that the kinetic energy

$$M v^2 \sim \varrho_1 r^3 \frac{r^2}{t^2} \sim \varrho_1 \frac{r^5}{t^2}$$

is deposited into the swept-up material. The energy of the material behind the shock is thus

$$\varrho_1 r^3 \dot{r}^2 = \varrho_1 \frac{r^5}{t^2} .$$

Equating this to the energy E, we find

$$\boxed{r = \left(\frac{E t^2}{\varrho_1}\right)^{1/5}} ,$$

which is exactly the scaling derived earlier, and simply expresses energy conservation within the shock.

$\Rightarrow$ energy-conserving phase

- We now know the velocity, radius, pressure and density at the shock, which are completely determined by E and ρ_1.

Supernova phases (see slides)

- **Energy-driven phase** (energy-conserving phase)
- With time, the compressed gas behind the shock will cool (radiate) and can thus increase its density further $\Rightarrow$ **formation of a dense shell**.
- This shell collects more material from the ambient medium (ρ_1) $\Rightarrow$ **„Snow plow phase"**.
- Finally, the **expansion is driven by momentum conservation** (by the inertia of the shell), until it comes to a halt, when the momentum of the swept-up material equals the initial momentum of the shell.

  $\Rightarrow$ **Momentum-driven phase**