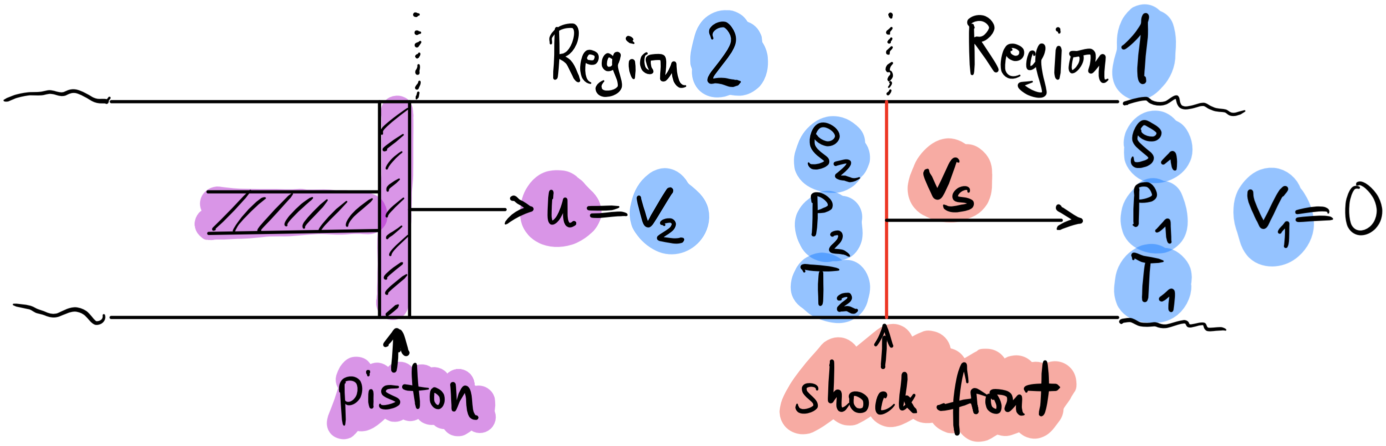
Derivation of the Shock Speed

- Let's consider a fluid pipe with a piston, which remains at rest at $x=0$ at $t=0$, and is then instantly accelerated to a velocity u into the positive x-direction.
- A discontinuity forms at $t=0$, which propagates with a velocity v_s to the right. There exists a region ahead of the shock, where the density, pressure, and the temperature still have their original values: ρ_1, P_1, T_1.
- In the region between the shock and the piston, the gas moves with the velocity u of the piston.



- The velocity difference between the two regions is u.
- In order to use the shock jump conditions derived before, we must transform into the coordinate frame where the shock is at rest (primed coordinates):

$$v_1' = -v_s \quad ; \quad v_2' = u - v_s = u + v_1'$$

and the velocity difference remains the same: $v_2' - v_1' = u$.

- Shock jump conditions:

$$\left.\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{v_1}{v_2} = \frac{(\gamma+1)\mathcal{M}^2}{(\gamma-1)\mathcal{M}^2+2} \\[2em] \frac{P_2}{P_1} &= \frac{1-\gamma+2\gamma\mathcal{M}^2}{\gamma+1} \end{aligned}\right\} \text{eliminate } \mathcal{M}^2$$

$$\Longrightarrow \boxed{\frac{\rho_2}{\rho_1} = \frac{P_2(\gamma+1)+P_1(\gamma-1)}{P_2(\gamma-1)+P_1(\gamma+1)} \qquad (1)}$$

- The matter current (flux) is $j = \rho_1 v_1'$, thus,

$$j^2 = \rho_1^2 v_1'^2 = \rho_2^2 v_2'^2 \quad \text{(mass conservation)},$$

and therefore:

$\rho_1 v_1'^2 + P_1 = \rho_2 v_2'^2 + P_2$
(momentum conservation

$$j^2 = \frac{1}{2}\left(\rho_1^2 v_1'^2 + \rho_2^2 v_2'^2\right) =$$

$$= \frac{1}{2}\left[j^2 + (P_2 - P_1)\rho_1 + \rho_1 \overbrace{\rho_2 v_2'^2}^{\rho_1/\rho_2\, j^2} \right]$$

• We thus obtain :

$$\boxed{\dot{j}^{2} = \frac{P_1 - P_2}{\rho_1 - \rho_2}\,\rho_1 \rho_2 \qquad (2)}$$

• The velocity difference can be written as

$$v_2' - v_1' = u = \frac{\rho_1}{\rho_1}\left(v_2' - v_1'\right) = \frac{1}{\rho_1}\left(\frac{\rho_1}{\rho_2}\underbrace{\rho_2 v_2'}_{\dot{j}} - \underbrace{\rho_1 v_1'}_{\dot{j}}\right)$$

$$= \underbrace{\dot{j}\,\underbrace{\frac{\rho_1 - \rho_2}{\rho_1 \rho_2}}_{(2^*)}}\ \overset{(2)}{=}\ \underbrace{\left[\frac{(P_1 - P_2)(\rho_1 - \rho_2)}{\rho_1 \rho_2}\right]^{1/2}}$$

• According to Eq. (1), the density ratio in (2^*) can be written as

$$\underbrace{\frac{\rho_1 - \rho_2}{\rho_1 \rho_2}}_{(2^{**})} = \frac{1}{\rho_1}\left(\frac{\rho_1}{\rho_2} - 1\right) \overset{(1)}{=} \frac{1}{\rho_1}\left[\frac{P_1(\gamma+1) + P_2(\gamma-1)}{P_1(\gamma-1) + P_2(\gamma+1)} - 1\right]$$

$$= \frac{2}{\rho_1}\left[\frac{P_1 - P_2}{P_1(\gamma-1) + P_2(\gamma+1)}\right]$$

• This back into u :

$$u = \left(\frac{P_1}{\rho_1}\right)^{1/2} \left(1 - \frac{P_2}{P_1}\right) \left[\frac{2}{\gamma - 1 + \frac{P_2}{P_1}(\gamma+1)}\right]^{1/2}$$

- Now we define the pressure ratio $\xi = \frac{P_2}{P_1}$ and the sound speed in the pre-shock region 1:

$$c_s = \left(\gamma \frac{P_1}{\rho_1}\right)^{1/2} \quad , \text{ which gives}$$

$$\boxed{u = \frac{c_s}{\gamma^{1/2}} \left|1 - \underset{>1}{\xi}\right| \left[\frac{2}{(\gamma-1) + \xi(\gamma+1)}\right]^{1/2}} \quad (2^{***})$$

- This can be rewritten as a quadratic eq. for the pressure ratio (ξ):

$$\xi^2 - \xi\left[2 + \frac{\gamma(\gamma+1)\,u^2}{2c_s^2}\right] + \left[1 - \frac{\gamma(\gamma-1)\,u^2}{2c_s^2}\right] = 0 \, ,$$

which has the solutions:

$$\xi = 1 + \frac{\gamma(\gamma+1)u^2}{4c_s^2} \pm \left[\left(1 + \frac{\gamma(\gamma+1)u^2}{4c_s^2}\right)^2 + \frac{\gamma(\gamma-1)u^2}{2c_s^2} - 1\right]^{1/2}$$

- The pressure ratio must be greater than 1, which excludes the negative branch of the solution $\Rightarrow$

$$\boxed{\xi = 1 + \frac{\gamma(\gamma+1)u^2}{4c_s^2} + \frac{\gamma u}{c_s}\left(1 + \frac{(\gamma+1)^2 u^2}{16 c_s^2}\right)^{1/2}}$$

(Note that as $u=0 \Rightarrow \xi = 1$)

- If $u \gg c_s$ (strong shock), then

$$\xi \approx \frac{\gamma(\gamma+1)}{2c_s^2} u^2 \qquad (3)$$

- Finally, we compute the shock speed $V_s = -v_1'$ as a function of ξ and u.
- From (2^*) together with (2^{**}) we get:

$$u = j\,\frac{\rho_1-\rho_2}{\rho_1\rho_2} = \underbrace{\rho_1 v_1'}_{j}\;\frac{2}{\rho_1}\left[\frac{P_1-P_2}{P_1(\gamma-1)+P_2(\gamma+1)}\right]$$

$$\overset{\xi=\frac{P_2}{P_1}}{=} 2v_1'\left[\frac{1-\xi}{\gamma-1+\xi(\gamma+1)}\right]$$

$$\Rightarrow v_1' = \frac{1}{2}u\,\frac{\gamma-1+\xi(\gamma+1)}{1-\xi}$$

$$\overset{(2^{***})}{=} \frac{1}{2}\left(\frac{2}{\gamma}\right)^{1/2} c_s\,|1-\xi|\,\frac{[\gamma-1+\xi(\gamma+1)]^{1/2}}{-|1-\xi|}$$

$$= -\frac{c_s}{(2\gamma)^{1/2}}\left[\gamma-1+\xi(\gamma+1)\right]^{1/2}$$

- And because $V_s = -v_1'$:

$$\boxed{V_s = \frac{c_s}{(2\gamma)^{1/2}}\left[\gamma-1+\xi(\gamma+1)\right]^{1/2} \quad \text{(Shock speed)}}$$

- Finally, let's consider the limit of a very strong shock ($u \gg c_s \Rightarrow \xi \gg 1$):

$$V_s \approx \frac{c_s}{(2\gamma)^{1/2}} \left[\xi (\gamma + 1) \right]^{1/2}$$

$$\overset{\xi \text{ from (3)}}{\approx} \frac{\cancel{c_s}}{(2\cancel{\gamma})^{1/2}} \frac{\cancel{\gamma^{1/2}} (\gamma+1)}{2^{1/2} \cancel{c_s}} u$$

$$\Longrightarrow \boxed{V_s \approx \frac{\gamma+1}{2} u > u}$$

(shock speed for strong shocks as a function of the piston speed u)

- For example, for $\gamma = 5/3$, the shock moves about 33% faster than the piston.

- Test with hydro 1D code (try out different u and γ)