Bondi accretion (Bondi 1952)

- Counterpart to Parker wind ($\tilde{v}=0$ @ $\tilde{r}=0$), but here we have inflows (accretion) ($\tilde{v}=0$ @ $r \to \infty$)

- Consider extended gaseous medium with pressure P_∞ and density ρ_∞ (at $r \to \infty$)

- Consider further a point mass M_* at rest inside this medium (BTW: if it were moving, it's the case of „Bondi-Hoyle accretion")

- As with Parker wind, we work in spherical coordinates and assume radial symmetry with $\vec{v} = v(r)\,\hat{e}_r$

- Stationary (steady state) conditions: $\frac{\partial}{\partial t} = 0$

- From the continuity equation, we already know (see Parker wind) that the accretion rate must be constant:

$$\mathring{\dot{M}} = -4\pi r^2 \rho v$$

(minus sign because of inflow instead of wind)

- Contrary to the Parker wind derivation, we will use a polytropic EOS here:

$$\boxed{P = K\varrho^{\Gamma}} \qquad (K = \text{const and polytropic index } \Gamma)$$

$\rightarrow$ sound speed: $\quad c_s^2(r) = \dfrac{dP}{d\varrho} = \Gamma K \varrho^{\Gamma-1}$

or using the continuity eq.: $\quad c_s^2 = \Gamma K \left(\dfrac{\dot{M}}{4\pi r^2 |v|}\right)^{\Gamma-1}$

- The Euler eq. in spherical symmetry (stationary):

$$\boxed{v\frac{dv}{dr} = -\frac{1}{\varrho}\frac{dP}{dr} - \frac{GM_*}{r^2}}$$

We now consider the isothermal limit ($\Gamma = 1$) and then the more general case ($1 < \Gamma < 5/3$).

Isothermal case ($\Gamma = 1$):

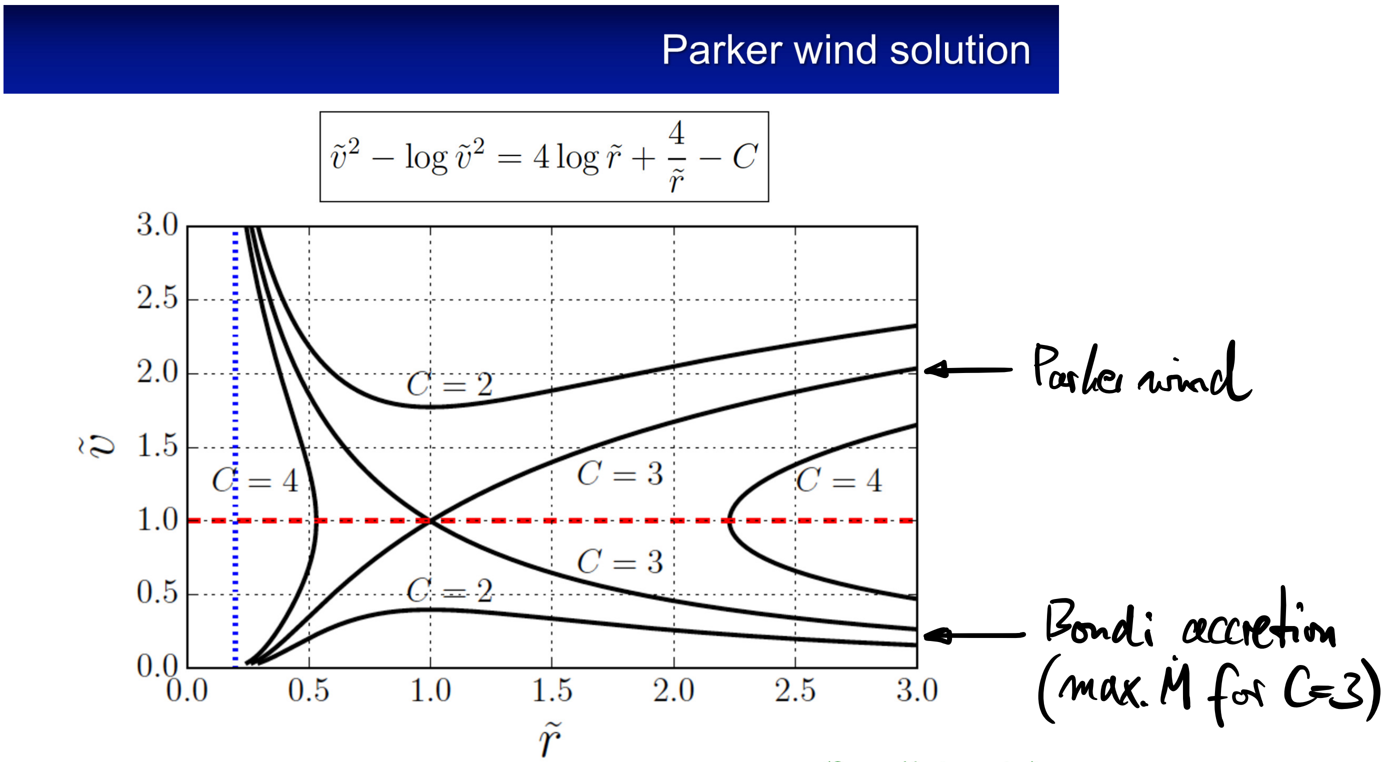
- $K = c_s^2$
- everything is identical to the Parker wind derivation, except that $\tilde{v} < 0$ for accretion (since $\tilde{v}$ appears quadratically in the solution, this accretion case is automatically included).

$$\boxed{\tilde{v}^2 - \ln(\tilde{v}^2) = 4\ln\tilde{r} + \frac{4}{\tilde{r}} - C}$$

with $\tilde{v} = \dfrac{v}{c_s} = \dfrac{v}{K^{1/2}}$ and $\tilde{r} = \dfrac{r}{r_s}$

with $r_s = \dfrac{GM_*}{2K}$ (sonic radius)

$\Rightarrow$ Same graphical solution as for Parker wind



(Sormani lecture notes)

Bondi accretion solutions are the ones with $C \leq 3$, because v_∞ must be zero.

Therefore the maximum accretion rate is given by $C=3$ and the actual value of $\dot{M}_{max}$ is obtained by inserting the physical variables (not the dimensionless) in the limit $r \to \infty$, i.e., at the boundary condition of the problem.

<u>In general</u>: $$\dot{M} = -4\pi \rho_\infty \lim_{r \to \infty} (r^2 v)$$

Take the $\lim_{r \to \infty}$ in the Parker wind solution:

$$\tilde{v}^2 = e^C \, \tilde{r}^{-4}$$

Inserting this into $\dot{M}$ yields:

$$\boxed{\dot{M} = 4\pi \rho_\infty e^{C/2} \cdot c_s \cdot r_s^2 = \pi \rho_\infty e^{C/2} \frac{(GM_*)^2}{c_s^3}}$$

(max. $\dot{M}$ for $C=3$)

Polytropic EOS case $\quad (1 < \Gamma < 5/3)$

- $P = K\rho^\Gamma \quad ; \quad c_s^2 = \frac{dP}{d\rho} = \Gamma K \rho^{\Gamma-1}$
- Starting again from the Euler equation:

$$v \frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM_*}{r^2}$$

- Integrate over r:

$$\frac{v^2}{2} + \underbrace{\int \frac{1}{\rho}\frac{dP}{dr}\,dr}_{(*)} - \frac{GM_*}{r} = \text{const.} \equiv C$$

(integration constant)

- Deal with the $(*)$ term:

$$\int \frac{1}{\rho}\frac{dP}{dr}\,dr = \int \frac{1}{\rho} K \frac{d\rho^\Gamma}{dr}\,dr = K\int \frac{1}{\rho}\frac{d\rho^\Gamma}{d\rho}\frac{d\rho}{dr}\,dr =$$

$$= K\int \frac{1}{\rho}\Gamma \rho^{\Gamma-1}\frac{d\rho}{dr}\,dr =$$

$$= K\Gamma \int \rho^{\Gamma-2} \frac{d\rho}{dr}\, dr = K\Gamma \int \frac{d}{dr}\left(\frac{\rho^{\Gamma-1}}{\Gamma-1}\right) dr$$

$$= \frac{K\Gamma}{\Gamma-1}\rho^{\Gamma-1} = \frac{c_s^2}{\Gamma-1}$$

· This back into the Euler eq.:

$$\boxed{\frac{v^2}{2} + \frac{c_s^2}{\Gamma-1} - \frac{GM_*}{r}} = C$$

· Since $v \to 0$ for $r \to \infty$, the constant must be

$$C = \frac{c_s^2(\infty)}{\Gamma-1}$$

· And at the sonic point (r_s) we must have

$$v = c_s \quad \text{at} \quad r = r_s = \frac{GM_*}{2c_s^2(r_s)}$$

$$\Rightarrow \frac{c_s^2(\infty)}{\Gamma-1} = \boxed{\frac{c_s^2(r_s)}{2} + \frac{c_s^2(r_s)}{\Gamma-1} - 2c_s^2(r_s)}$$

$$\Rightarrow c_s^2(r_s) = c_s^2(\infty)\left[\frac{1}{(\Gamma-1)\left(\frac{1}{2}+\frac{1}{\Gamma-1}-2\right)}\right]$$

$$\boxed{\Rightarrow c_s(r_s) = c_s(\infty)\left[\frac{2}{5-3\Gamma}\right]^{1/2}}$$

- Because of the proportionality $c_s^2 \sim \rho^{\Gamma-1}$, the density is
$$\rho(r_s) = \rho_\infty \left[\frac{c_s(r_s)}{c_s(\infty)}\right]^{\frac{2}{\Gamma-1}}$$

- This also specifies the accretion rate:
$$\dot{M} = 4\pi r^2 \rho v = \text{const} = 4\pi r_s^2\, \rho(r_s)\, c_s(r_s)$$

and thus,
$$\boxed{\dot{M}} = 4\pi r_s^2 \rho_\infty \left(\frac{2}{5-3\Gamma}\right)^{\frac{1}{\Gamma-1}} c_s(\infty) \left(\frac{2}{5-3\Gamma}\right)^{1/2}$$
$$\boxed{= \pi G^2 M_*^2 \frac{\rho_\infty}{c_s^3(\infty)} \left(\frac{2}{5-3\Gamma}\right)^{\frac{5-3\Gamma}{2(\Gamma-1)}}}$$
(maximum accretion rate for polytropic case)

Compare max. accretion rates for isothermal and polytropic cases:

$$\dot{M}_{max}^{\text{isothermal}} = \pi \rho_\infty (GM_*)^2 c_s^{-3}(\infty)\, \boxed{e^{3/2}} \quad (\Gamma \to 1)$$

$$\dot{M}_{max}^{\text{polytropic}} = \pi \rho_\infty (GM_*)^2 c_s^{-3}(\infty) \boxed{\left(\frac{2}{5-3\Gamma}\right)^{\frac{5-3\Gamma}{2(\Gamma-1)}}}$$

Isothermal limit

$$\left[\begin{aligned} &\text{Def: } n = \frac{5-3\Gamma}{2(\Gamma-1)} \\ &\Rightarrow 2n\Gamma - 2n + 3\Gamma = 5 \\ &\quad \Gamma = \frac{5+2n}{3+2n} \\ &\Rightarrow \frac{2}{5-3\Gamma} = 1 + \frac{3}{2n} \\ &\Rightarrow \lim_{n\to\infty}\left(1+\frac{3}{2n}\right)^n = e^{3/2} \end{aligned}\right]$$

$\longrightarrow$ see graphical comparison on the course slides.

Side note:

if we want the solution for $v(r)$, we get that from the cont. eq. $-4\pi r^2 \rho v = \dot{M} \Longrightarrow$

$$v = \frac{-\dot{M}}{4\pi r^2 \rho(r)} = \frac{-\dot{M}}{4\pi r^2 \rho_\infty} \left(\frac{c_s(\infty)}{c_s(r)} \right)^{\frac{2}{\Gamma-1}}$$

... and this back into Euler eq. gives $v(r)$.