Parker wind solution

Parker (1958) : winds from stars (massive) $\longrightarrow$ radial motion

- Speeds of ~1000 km/s
- make cometary tails
- impacting Earth's magnetic field
- halting at ~100 AU distance from Sun by the pressure of the interstellar medium (ISM)

Necessity for radial motions

Star in spherical symmetry and in vacuum
$\longrightarrow$ there must be radial outward motion

Hydrostatic (v=0) equilibrium : (from Euler eq.)

also assume isothermal gas $P = c_s^2 \rho$

$$0 = -\frac{c_s^2}{\rho}\frac{d\rho}{dr} - \frac{GM_*}{r^2}$$

$$\Rightarrow \int \frac{d\rho}{\rho} = -\frac{GM_*}{c_s^2}\int\frac{dr}{r^2} \Rightarrow \rho(r) = \rho_0\, e^{\frac{GM_*}{c_s^2 r}}$$

with $\rho(r\to\infty) = \rho_0$

ρ_0 is the background density for $r \to \infty$ $\Rightarrow$
the pressure at $r \to \infty$ does not vanish.
Hence, without a confinement pressure, there must be radial motion outwards from the star. $\Rightarrow$ wind

Although we assumed an isothermal EOS, Parker showed that the same holds for non-isothermal EOS.

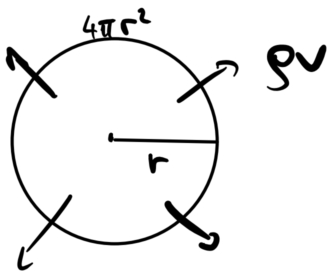
Parker wind solution

- assume spherical symmetry: all variables are functions of radius (r) only
- assume stationary flows: $\partial / \partial t = 0$
- Velocity field: $\mathbf{v} = v(r)\, \hat{e}_r$

In steady state, the mass outflow rate is given by

$$\dot{M} = \underbrace{4\pi r^2}_{\text{surface}} \underbrace{\rho v}_{\text{mass flux}}$$

(c.f. derivation of the hydro equations)



$\boxed{\dot{M} = \text{const.}}$ and independent of radius.

We can show this by using the continuity eq. in spherical and stationary state:

$$\underbrace{\cancel{\frac{\partial}{\partial t}}}_{\text{stationary}} \rho + \nabla \cdot (\rho \vec{v}) = 0$$

div operator in spherical coordinates:

$$\operatorname{div} \vec{A} = \nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \cancel{\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta)}$$

$$+ \cancel{\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}} \quad \text{(spherical symmetry)}$$

$$\Rightarrow \nabla \cdot (\rho\vec{v}) \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0$$

$$\Rightarrow r^2 \rho v = \text{const.}$$

Pick $\dot{M}/4\pi$ as the const., which gives

$$\boxed{\dot{M} = 4\pi r^2 \rho v = \text{const.}} \quad \text{(Mass outflow rate)}$$

Now our goal is to find $v(r)$. So we write down the Euler eq. in spherical, steady flow conditions:

$$\cancel{\frac{\partial v}{\partial t}} + v\frac{dv}{dr} = -\frac{c_s^2}{\rho}\frac{d\rho}{dr} - \frac{GM_*}{r^2}$$

steady

Now we want to eliminate ρ from this eq., so we make use of the continuity eq.:

$$r^2\rho v = \text{const.} \Rightarrow \frac{d}{dr}(r^2\rho v) = 0$$

$$\Rightarrow \rho v\, 2r + r^2 v\frac{d\rho}{dr} + r^2\rho\frac{dv}{dr} = 0 \quad \Big|\, / r^2\rho v$$

$$\Rightarrow \frac{2}{r} + \frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{v}\frac{dv}{dr} = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} = -\left(\frac{1}{v}\frac{dv}{dr} + \frac{2}{r}\right)$$

... and set this into the Euler eq. above:

$$v\frac{dv}{dr} = c_s^2\left(\frac{1}{v}\frac{dv}{dr} + \frac{2}{r}\right) - \frac{GM_*}{r^2}$$

$$\Rightarrow v\frac{dv}{dr} - \frac{c_s^2}{v}\frac{dv}{dr} = \frac{2c_s^2}{r} - \frac{GM_*}{r^2}$$

$$\Rightarrow \frac{dv}{dr}\left(v - \frac{c_s^2}{v}\right) = \frac{2c_s^2}{r}\left(1 - \frac{1}{r}\underbrace{\frac{GM_*}{2c_s^2}}_{= r_s}\right)$$

(sonic radius)

This allows us to define dimensionless variables:

$$\tilde{r} = \frac{r}{r_s} \qquad \text{and} \qquad \tilde{v} = \frac{v}{c_s} \quad \text{(Mach number)}$$

Insert this into above Euler eq.:

$$\frac{d\tilde{v}}{d\tilde{r}} \frac{\cancel{c_s}}{\cancel{r_s}} \left(\tilde{v} \cancel{c_s} - \frac{\cancel{c_s^2}}{\tilde{v} \cancel{c_s}} \right) = \frac{2 \cancel{c_s^2}}{\tilde{r} \cancel{r_s}} \left(1 - \frac{1}{\tilde{r}} \right)$$

$$\frac{d\tilde{v}}{d\tilde{r}} \left(\tilde{v} - \frac{1}{\tilde{v}} \right) = \frac{2}{\tilde{r}} \left(1 - \frac{1}{\tilde{r}} \right) \quad \text{(much simpler and dimensionless)}$$

Now we study the solutions of this equation.

Integrate:

$$\frac{1}{2} \tilde{v}^2 - \ln(\tilde{v}) = 2 \ln(\tilde{r}) + 2 \frac{1}{\tilde{r}} - \text{Const}$$

(integration constant)

$$\Rightarrow \boxed{\tilde{v}^2 - \ln(\tilde{v}^2) = 4 \ln(\tilde{r}) + \frac{4}{\tilde{r}} - C}$$

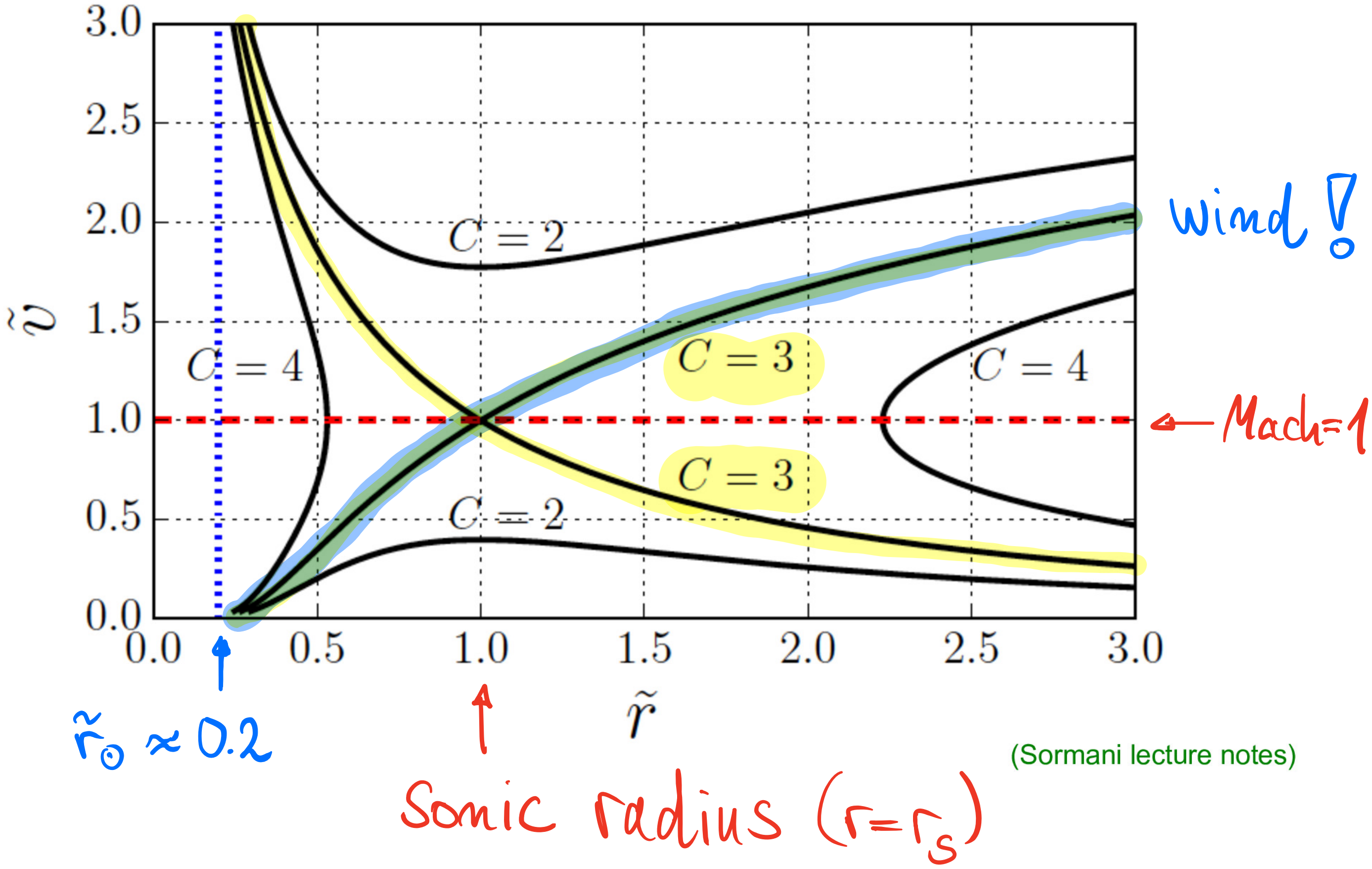
Parker wind solution

And now we have to find C.

In order to find C, we plot the wind solution for different values of C.

Parker wind solution

$$\boxed{\tilde{v}^2 - \log \tilde{v}^2 = 4 \log \tilde{r} + \frac{4}{\tilde{r}} - C}$$



The only physically plausible solutions are the two branches with $C=3$, because these solutions obey our basic approach/condition that $v(r_s) = c_s$ (go through sonic point).

Now we have to see which of the two remaining solutions for C=3 is the correct one. Let's consider the sonic radius for the Sun:

$$r_s = \frac{GM_*}{2c_s^2} \approx 5 R_\odot$$

with $c_s = \left(\frac{k_B T}{\mu m_H}\right)^{1/2} \approx 140\ \mathrm{km/s}$

($T \approx 1.5 \cdot 10^6\,\mathrm{K}$ in the solar corona)

$$\Downarrow$$

$$\tilde{r}_\odot = \frac{R_\odot}{r_s} \approx 0.2$$