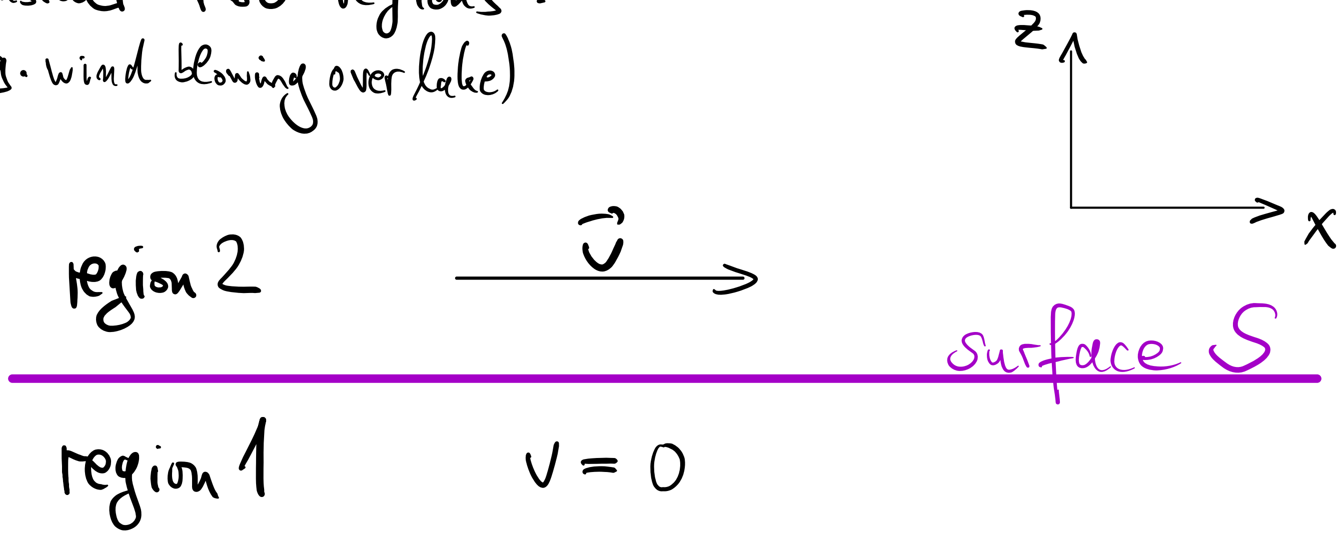
Kelvin-Helmholtz instability

Consider two regions:
(e.g. wind blowing over lake)



Now start with Euler equation:

$$\frac{D}{Dt}\vec{v} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho}$$

Take div on both sides and assume $\rho = \text{const}$ ($\text{div}\,\vec{v} = 0$)

$$\Rightarrow \nabla^2 P = 0.$$

The ansatz $\delta P = f(z)\, e^{i(k_x x - \omega t)}$

leads to an oscillator eq. ($k = k_x$):

$$\frac{\partial^2 f(z)}{\partial z^2} = k^2 f(z)$$

This has solutions $f(z) \sim e^{\pm k z}$.

In region 2 (above, where $v \neq 0$), i.e. for $z>0$, the exponentially growing solution $\sim e^{+kz}$ is ruled out physically, because it diverges at large z.
Thus, the pressure perturbation must be

$$\delta P_2 \sim e^{-kz} e^{i(kx-\omega t)}$$

above the surface (region 2).

To 1st order in the velocity perturbations δv_z, Euler's eq. reads (in region 2):

$$\frac{\partial(\delta v_z)}{\partial t} + v \frac{\partial(\delta v_z)}{\partial x} = k \frac{\delta P_2}{\rho_2}$$

$$\left(\text{from } (\vec{v}\cdot\nabla)\vec{v} = (v_x \partial_x + v_z \partial_z) \begin{pmatrix} \delta v_x \\ \delta v_z \end{pmatrix} \right)$$

Again the ansatz $\delta v_z \sim e^{i(kx-\omega t)}$ leads to

$$\boxed{\delta v_z = \frac{k\,\delta P_2}{i\rho_2 (kv-\omega)} \quad \text{in Region 2} \quad \text{Eq. (1)}}$$

Now for region 1, we have (in analogy to above, but with $\delta P_1 \sim e^{+kz} e^{i(kx-\omega t)}$

$$\frac{\partial(\delta v_z)}{\partial t} = -k\,\frac{\delta P_1}{\rho_1}$$

$$\Rightarrow \boxed{\delta v_z = \frac{k\,\delta P_1}{i\rho_1\,\omega} \qquad \text{in Region 1} \quad \text{Eq.(2)}}$$

Now we consider the boundary surface S (the z coordinate of the surface)
and how it changes its velocity in z

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \underbrace{(\vec{v}\cdot\nabla)}_{\text{here just } v\partial_x \text{ as before}} S = \delta v_z \qquad \left(\begin{array}{c}\text{Region 2}\\ v \neq 0\end{array}\right)$$

Once again assuming plane waves: $S \sim e^{i(kx-\omega t)}$,
we get

$$-i\omega S + vik\,S = \delta v_z$$

$$\Rightarrow \delta v_z = i(kv-\omega)\,S$$

Now set this into Eq.(1):

$$\delta P_2 = i^2 \rho_2 (kv-\omega)^2\, S \cdot \frac{1}{k}$$

$$\boxed{= -\frac{\rho_2 S}{k}(kv-\omega)^2 \qquad \text{Region 2} \quad \text{Eq.(3)}}$$

Equivalently for region 1:

$$-i\omega S + 0 = \delta v_z \quad (v=0 \text{ in region } 1)$$

$$\Rightarrow \delta v_z = -i\omega S$$

This into Eq. (2), we get:

$$\boxed{\delta P_1 = \frac{\rho_1 S}{k}\omega^2 \qquad \text{Region 1} \quad \text{Eq. } \boxed{4}}$$

Finally, there must be pressure balance at the boundary:

$$\underset{(\text{Eq. } 4)}{\delta P_1} \overset{!}{=} \underset{(\text{Eq. } 3)}{\delta P_2}$$

$$\Rightarrow \rho_1 \omega^2 = -\rho_2 (kv-\omega)^2$$

$$\Rightarrow \rho_1 \omega^2 + \rho_2 \omega^2 - 2\rho_2 kv\omega + \rho_2 k^2 v^2 = 0$$

$$\Rightarrow (\rho_1+\rho_2)\,\omega^2 - 2\rho_2 kv\,\omega + \rho_2 k^2 v^2 = 0$$

Quadratic eq. in ω, and the solution is

$$\omega = \frac{2\rho_2 kv \pm \sqrt{4\rho_2^2 k^2 v^2 - 4(\rho_1+\rho_2)\rho_2 k^2 v^2}}{2(\rho_1+\rho_2)}$$

$$= \boxed{\frac{kv}{\rho_1+\rho_2}\left(\rho_2 \pm i\sqrt{\rho_1\rho_2}\right)}$$

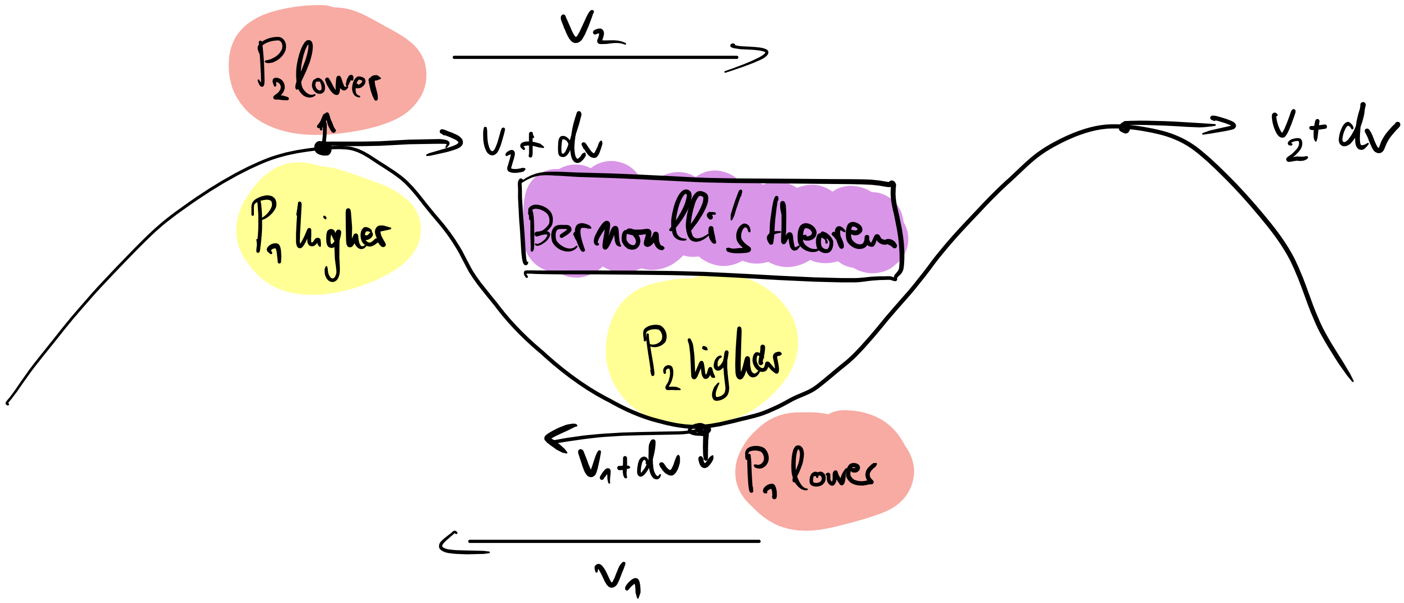
For $\rho_1, \rho_2 > 0$

ω is imaginary

$\Rightarrow$ Kelvin-Helmholtz instability

Intuitive explanations for the KH instability

1. Imagine two flows going in opposite directions



2. Another way: Imagine two packets of fluid, one with v, another with $v + dv$, and they mix.

	Region 1	Region 2
velocities before mixing	v	$v + dv$
velocities after mixing	$v + \frac{1}{2} dv$	$v + \frac{1}{2} dv$

Because of momentum conservation:

$$v^{new} = v + \frac{1}{2} dv = \frac{mv + m(v+dv)}{2m}$$

Now calculate the change in specific kinetic energy:

$$de_{kin} = e_{kin}^{old} - e_{kin}^{new}$$
$$= \frac{1}{2}\left[v^2 + (v+dv)^2\right] - \frac{1}{2}\left[2\left(v + \frac{1}{2} dv\right)^2\right]$$

$$= \frac{1}{4}\, d v^2 > 0$$

$\Rightarrow$ Mixing releases kin. energy, which grows the instability further.

Time scale for KH instability

Remember: $\omega = \frac{k v}{\rho_1 + \rho_2}\left(\rho_2 + i\sqrt{\rho_1 \rho_2}\right)$

$$\boxed{\tau_{KH} = \left[\mathrm{Im}(\omega)\right]^{-1} = \frac{\rho_1 + \rho_2}{k v \sqrt{\rho_1 \rho_2}}}$$

(Time scale for KH instability)

Examples: if $\rho_1 = \rho_2 \Rightarrow \tau \sim \frac{2\rho_1}{\rho_1} \sim 2$

if $\rho_1 = 10\rho_2 \Rightarrow \tau \sim \frac{11\rho_2}{\sqrt{10}\,\rho_2} \sim 3.5$

- higher density contrast $\Rightarrow$ slower growth
- higher velocity contrast $\Rightarrow$ faster growth
- Smallest scales (highest k) grow the fastest