Classical Jeans instability

(gravitational instability)
(Jeans 1902)

Hydro Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad (1)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi \qquad (2)$$

Now assume static background + small perturbations:

$\rho \to \rho_0 + \delta\rho$ with $\rho_0 = \text{const.}$

$\vec{v} \to \cancel{\vec{v}_0} + \delta\vec{v}$ with $\vec{v}_0 = 0$.

$P \to P_0 + \delta P$ with $P_0 = \text{const.}$

$\Phi \to \Phi_0 + \delta\Phi$ with $\Phi_0 = \text{const.}$

in (1) and (2):

$$\frac{\partial(\delta\rho)}{\partial t} + \nabla \cdot \left[(\rho_0 + \delta\rho) \cdot \delta\vec{v} \right] = 0$$

$$\underset{\text{(linear)}}{\Longrightarrow} \boxed{\frac{\partial(\delta\rho)}{\partial t} + \rho_0 (\nabla \cdot \delta\vec{v}) = 0} \qquad (1)'$$

$$\frac{\partial(\delta\vec{v})}{\partial t} + \cancel{(\delta\vec{v}\cdot\nabla)\,\delta\vec{v}}^{\to 0} = -\frac{1}{\rho_0+\cancel{\delta\rho}^{\to 0}}\nabla(\delta P) - \nabla(\delta\Phi)$$

(2nd order)

$$\Longrightarrow \boxed{\frac{\partial(\delta\vec{v})}{\partial t} = -\frac{1}{\rho_0}\nabla(\delta P) - \nabla(\delta\Phi)} \quad (2)'$$

Take $\frac{\partial}{\partial t}(1)'$ and $\rho_0\nabla\cdot(2)'$:

$$\frac{\partial^2(\delta\rho)}{\partial t^2} + \rho_0\left(\nabla\cdot\frac{\partial\delta\vec{v}}{\partial t}\right) = 0 \qquad (1)''$$

$$\rho_0\cdot\left(\nabla\frac{\partial(\delta\vec{v})}{\partial t}\right) = -\nabla^2(\delta P) - \rho_0\nabla^2(\delta\Phi) \qquad (2)''$$

$(2)''$ in $(1)''$:

$$\boxed{\frac{\partial^2(\delta\rho)}{\partial t^2} - \nabla^2(\delta P) - \rho_0\nabla^2(\delta\Phi) = 0} \quad (3)$$

Now assume adiabatic perturbations: $\boxed{\delta P = c_s^2\,\delta\rho}$ (4)

$c_s =$ const. (isothermal)

And we add Poisson Equation: $\boxed{\nabla^2(\delta\Phi) = 4\pi G\,\delta\rho}$ (5)

(4) and (5) in (3):

$$\boxed{\frac{\partial^2(\delta\rho)}{\partial t^2} - c_s^2\,\Delta(\delta\rho) - 4\pi G\rho_0\,\delta\rho = 0} \quad (6)$$

(form of a wave equation)

We can insert plane waves: $\delta\rho = A\cdot e^{i(k\vec{x}-\omega t)}$

$\left[\frac{\partial^2}{\partial t^2} \to -\omega^2 \; ; \; \nabla^2 \to -k^2\right]$

Into (6): $-\omega^2 + c_s^2 k^2 - 4\pi G\rho_0 = 0$

$$\Rightarrow \boxed{\omega^2 = c_s^2\left(k^2 - k_J^2\right)}$$

with the Jeans wavenumber $\boxed{k_J = \left(\frac{4\pi G\rho_0}{c_s^2}\right)^{1/2}}$

Normal waves if $k > k_J$ are stable.

However, we have exponential growth if $k < k_J$

$\Rightarrow \omega^2 < 0$ (instability criterion)

We can also define a Jeans length

$$\boxed{\lambda_J = \frac{2\pi}{k_J} = \left(\frac{\pi c_s^2}{G\rho_0}\right)^{1/2}}$$ Jeans length

If $\lambda > \lambda_J \Rightarrow$ instability

If $\lambda < \lambda_J \Rightarrow$ stable

We can also define a Jeans mass

$$\boxed{M_J = \frac{4\pi}{3}\left(\frac{\lambda_J}{2}\right)^3 \cdot \rho_0} \sim c_s^3 \cdot \rho_0^{-1/2}$$

$$\sim T^{3/2} \cdot \rho_0^{-1/2}$$

Implication: $T \uparrow \;\rightarrow\; M_J \uparrow$

$\rho \uparrow \;\rightarrow\; M_J \downarrow$

If mass of cloud $M > M_J \Rightarrow$ collapse.