Derivation of Energy Equation of Hydrodynamics

From the momentum eq. (velocity eq.):

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla P \quad \Big| \cdot \vec{v}$$

$$\vec{v}\cdot\left[\partial_t \vec{v} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\frac{1}{\rho}\vec{v}\cdot\nabla P$$

$$\frac{1}{2}\partial_t \vec{v}^2 + \frac{1}{2}(\vec{v}\cdot\nabla)\vec{v}^2 = -\frac{1}{\rho}\left(\nabla\cdot(\vec{v}P) - P(\nabla\cdot\vec{v})\right) \Big| \cdot\rho$$

$$\frac{1}{2}\rho\left[\partial_t v^2 + (\vec{v}\cdot\nabla)\vec{v}^2\right] + \underbrace{\frac{1}{2}\vec{v}^2\left[\partial_t\rho + \nabla\cdot(\rho\vec{v})\right]}_{=0 \text{ (cont. eq.)}} = P(\nabla\cdot\vec{v}) - \nabla\cdot(\vec{v}P)$$

$$\partial_t\left(\frac{1}{2}\rho\vec{v}^2\right) + \nabla\cdot\left(\frac{1}{2}\rho v^2\vec{v}\right) = P(\nabla\cdot\vec{v}) - \nabla\cdot(\vec{v}P)$$

$$\boxed{\partial_t(\rho \underset{\vec{v}^2/2}{\underset{\|}{e_{kin}}}) + \nabla\cdot\left[(\rho e_{kin} + P)\vec{v}\right] = P(\nabla\cdot\vec{v})} \quad (1)$$

Now from Lagrangian form of the Continuity Eq.:

$$\boxed{\nabla\cdot\vec{v} = -\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{D\log\rho}{Dt}} \quad (2)$$

Furthermore, we know from adiabatic gas:

$$P \sim \rho^\gamma \Rightarrow \log P \sim \gamma \log\rho \Rightarrow \boxed{\frac{D\log P}{Dt} = \gamma\frac{D\log\rho}{Dt}} \quad (3)$$

$(P = K\rho^\gamma)$ $\qquad (+\log K)$

Combine (2) and (3):

$$\nabla\cdot\vec{v} = -\frac{1}{\gamma}\frac{D\log P}{Dt} = -\frac{1}{\gamma P}\frac{DP}{Dt}$$

comoving deriv: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$

$$\overset{\text{comoving der.}}{=} -\frac{1}{\gamma P}\left(\partial_t P + (\vec{v}\cdot\nabla)P\right)$$

$$= -\frac{1}{\gamma P}\left(\partial_t P + \nabla\cdot(P\vec{v}) - P(\nabla\cdot\vec{v})\right)$$

Now we need to solve this for $\nabla\cdot\vec{v}$:

$$\left(1-\frac{1}{\gamma}\right)(\nabla\cdot\vec{v}) = -\frac{1}{\gamma P}\left[\partial_t P + \nabla\cdot(P\vec{v})\right]$$

$$\Rightarrow \boxed{\nabla\cdot\vec{v} = -\frac{1}{(\gamma-1)P}\left[\partial_t P + \nabla\cdot(P\vec{v})\right]} \quad (4)$$

Finally, put (4) into (1):

$$\partial_t\left(\rho e_{kin} + \frac{P}{\gamma-1}\right) + \nabla\cdot\left[\left(\rho e_{kin} + P + \frac{P}{\gamma-1}\right)\vec{v}\right] = 0$$

With $P = \rho e(\gamma-1)$ we get
(ideal gas EOS)

$$\boxed{\partial_t\left(\underbrace{\rho e_{kin} + \rho e}_{\rho e_{tot}}\right) + \nabla\cdot\left[\left(\underbrace{\rho e_{kin} + \rho e}_{\rho e_{tot}} + P\right)\vec{v}\right] = 0}$$

(Energy Equation)