Equation of State (EOS)

Recall:
- 5 hydro equations (mass, momentum, energy)
- But we have 6 variables ($\rho, \vec{v}, e_{tot}, P$)

$\Rightarrow$ Need a 6th eq. to close the system.

$\Rightarrow$ Need the **EOS** (which relates ρ, P, e)

Usually the EOS is that of an **ideal gas**

(point particles with perfect collisions)

Ideal gas equation: $PV = N k_B T$

(1) Using $V = \frac{m}{\rho}$ we get $\frac{P}{\rho} = \frac{N}{m} k_B T$

(2) Specific internal energy: $e = \frac{f}{2} \frac{N}{m} k_B T$

(3) adiabatic index: $\gamma = \frac{C_P}{C_V} = 1 + \frac{2}{f}$

$\Rightarrow P = \rho e (\gamma - 1)$

(f: degrees of freedom)

Also (2) and (3) implies: $e = \frac{1}{\gamma - 1} \left(\frac{N}{m}\right) k_B T$, with $\frac{N}{m} = \frac{1}{\mu m_H}$

This also implies adiabatic gas (entropy = const): $P \sim \rho^{\gamma}$

Speed of sound ("sound speed"):

$$c_s = \left(\frac{\partial P}{\partial \rho}\right)^{1/2} \qquad \text{(basic definition)}$$

For ideal gas: $P = \rho e (\gamma - 1) \Rightarrow \frac{\partial P}{\partial \rho} = e(\gamma - 1)$

and with $e = \frac{1}{\gamma - 1} \frac{k_B T}{\mu m_H}$, we get $\frac{\partial P}{\partial \rho} = \frac{k_B T}{\mu m_H}$

Thus, the sound speed of an ideal gas is

$$\boxed{c_s = \left(\frac{k_B T}{\mu m_H}\right)^{1/2}}$$

"mean particle weight"

for example:
- pure hydrogen (atomic): $\mu = 1$
- fully ionised H: $\mu = 0.5$
  (normal cosmic mixture with He, etc.: $\mu = 0.6$)
- molecular cloud: $\mu = 2.3$

Special cases of EOS:

Isothermal gas ($T = \text{const.}$)

c_s = const. (as long as μ = const.)

$\Rightarrow$ $P = c_s^2 \rho$ (isothermal gas pressure)

Also implies that $\gamma \to 1$ $(f \to \infty)$

Barotropic EOS: $P = P(\rho)$ (no T dependence)

For example "polytropic EOS": $P \sim \rho^{\Gamma}$

or $P = K \cdot \rho^{\Gamma}$ $\Rightarrow T \sim \rho^{\Gamma - 1}$

where Γ is the "polytropic index":

$\Gamma > 1$: heating when compressed

$\Gamma < 1$: cooling when compressed

$\Gamma = 1$: isothermal case (T = const)

→ examples, see slides...