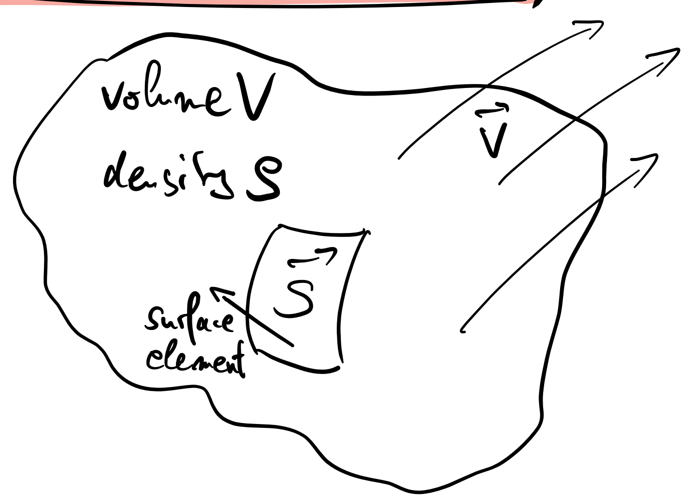
Astrophysical Gas Dynamics

Basic Equations of Hydrodynamics

- Introduce gas/fluid variables:
  density ρ, velocity $\vec{v}$, energy e, pressure P, temperature T ... are related by the Hydro eqs. and the Equation of State

Derive the „Continuity Equation" (mass conservation)

$$\text{mass} = \int \rho \, dV$$



(1) change of mass with time:

$$\frac{\partial}{\partial t} \int_V \rho \, dV$$

(2) mass flux through the surface: $-\int_S \rho \vec{v} \, d\vec{S}$

(1) and (2) must be equal!

$$\frac{\partial}{\partial t} \int_V \rho \, dV = - \int_S \rho \vec{v} \, d\vec{S}$$

Now use Gauss' theorem („divergence theorem"):

$$\int_S (\vec{q})\, d\vec{S} = \int_V (\nabla \cdot \vec{q})\, dV$$

$$\Rightarrow \frac{\partial}{\partial t} \int_V \rho\, dV = - \int_V \nabla \cdot (\rho \vec{v})\, dV$$

Now pull $\frac{\partial}{\partial t}$ into the integral:

$$\Rightarrow \int \frac{\partial}{\partial t} \rho\, dV = - \int \nabla \cdot (\rho \vec{v})\, dV$$

Drop integral over dV, because this must be true for any volume V:

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{v}) = 0}$$

Continuity Equation (Eulerian form).

Using the product rule, we get

$$\nabla\cdot(\rho\vec{v}) = \vec{v}\cdot(\nabla\rho) + \rho(\nabla\cdot\vec{v})$$

... and back into the Continuity Eq:

$$\Rightarrow \frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\rho = -\rho(\nabla\cdot\vec{v})$$

Define the co-moving (Lagrangian) derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\nabla)$$

cartesian
$$\nabla = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}$$

and use this in the above eq.:

$$\Rightarrow \boxed{\frac{D\rho}{Dt} = -\rho(\nabla\cdot\vec{v})}$$

Continuity Eq. in Lagrangian form.

Remark: for incompressible fluid: $\nabla\cdot\vec{v}=0$

Conservation of momentum

Basically follow the same approach as for mass, but now for momentum:

$$\frac{\partial}{\partial t}\int_V (\rho\vec{v})\,dV = \underbrace{-\int_S (\rho\vec{v})\,\vec{v}\,d\vec{S}}_{\text{momentum flux}}\ \underbrace{-\int_S P\,d\vec{S}}_{\text{pressure force}}$$

We can also write this as

$$\frac{\partial}{\partial t}\int_V (\rho\vec{v})\,dV = -\int_S \Big(\rho\underbrace{\vec{v}\vec{v}}_{\substack{\text{diadic}\\\text{product}\\(3\times3\text{ matrix})}} + \underbrace{\mathbb{1}}_{\text{unit matrix}}\cdot P\Big)\,d\vec{S}$$

$$\overset{\text{Gauss}}{=} -\int_V \nabla\cdot\left(\rho\vec{v}\vec{v} + \mathbb{1}P\right)dV$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\vec{v}) = -\nabla P$$

Now expand derivatives:

$$\rho\frac{\partial\vec{v}}{\partial t} + \underbrace{\vec{v}\,\frac{\partial\rho}{\partial t} + \vec{v}\,\nabla\cdot(\rho\vec{v})}_{=0\ (\text{continuity eq.})} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P$$

$$\Rightarrow \boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla P}{\rho}}$$

Euler equation (momentum conservation) in Eulerian form.

Using the Lagrangian derivative:

$$\boxed{\frac{D\vec{v}}{Dt} = -\frac{\nabla P}{\rho}}$$

Momentum Equation in Lagrangian form.

Conservation of Energy

Similar to mass and momentum eq., but for ρe_{tot}, which is the total energy density (means that e_{tot} is called specific (per unit mass) energy).
Also need to add source terms as for the momentum equation (work from contraction/expansion, i.e. pressure):

$$\boxed{\frac{\partial}{\partial t}(\rho e_{tot}) + \nabla \cdot \left[(\rho e_{tot} + P)\vec{v}\right] = 0}$$

Energy Equation in Eulerian form.

with $e_{tot} = \underbrace{e}_{\text{internal energy}} + \underbrace{v^2/2}_{\text{kinetic energy}}$ („specific" energies)

$$\boxed{\frac{D}{Dt}(e_{tot}) = -\frac{P}{\rho}(\nabla\cdot\vec{v}) - \frac{1}{\rho}\vec{v}\cdot\nabla P}$$

Energy Eq. in Lagrangian form.

Taken together, we have
5 coupled PDEs, but 6 variables
(1 mass, 3 momentum, 1 energy) $(\rho, \vec{v}, P, e_{tot})$

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$\Rightarrow$ System is not closed, and needs a closure eq.: the Equation of State (EOS)
(relates ρ, P, e)