Wave Solutions to Linearised Ideal MHD Equations

Ideal MHD equations are good 1^{st} order approximations to molecular clouds – the birth place of stars.

$\boxed{\text{Why?}}$

> $R_e \sim 10^9 \Rightarrow |\nu \nabla^2 v| \ll 1$ ← viscous dissipation

> $R_m \sim 10^{16} \Rightarrow |\eta \nabla^2 B| \ll 1$

$\Rightarrow$ ideal (w.r.t dissipation) ← magnetic resistivity

other non-ideal effects may be important

also approximately isothermal $\gtrsim 0.1\,\text{pc}$

with $\sigma_v / c_s = \mathcal{M} \sim 10$

$L \sim O(\text{pc}),\ T = L/\sigma_v \sim O(\text{Myr})$

$\text{Temp.} \sim 10\,\text{K},\quad n \sim 100\,\text{cm}^{-3}$

Hence, let's begin with seeking solutions to the ideal MHD eqs.

Ideal isothermal MHD equations

$$\frac{\partial \rho}{\partial t} + \underbrace{\nabla\cdot(\rho v)}_{\text{mass advection}} = 0 \qquad \text{continuity}$$

$$\frac{\partial \rho v}{\partial t} + \nabla\cdot(\rho v v) = \overbrace{-\nabla P}^{\text{thermal pressure}} + \underbrace{\frac{1}{4\pi}(\nabla\times B)\times B}_{\text{Lorentz force}} \qquad \text{momentum}$$

$$\frac{\partial B}{\partial t} = \underbrace{\nabla\times(v\times B)}_{\text{Faraday's + Ampere's + Ohm's law}} \qquad \text{induction}$$

$$\nabla\cdot B = 0 \quad \text{divergence-free constraint} \qquad P = c_s^2\rho \quad \text{EoS.}$$

Linearised variables:

Consider a plasma with a stationary, homogenous background with only small (linear) perturbations from the stationary state:

linear
· $X_0 \delta X$
· X_0^2
· $X_0, \delta X$

$$v = v_0 + \delta v(\vec{r},t) = \delta v \qquad \underbrace{\boxed{\text{Consider } v_0 = 0}}_{\text{no bulk motion}}$$

$$\rho = \rho_0 + \delta\rho(\vec{r},t)$$

$$B = B_0 + \delta B(\vec{r},t)$$

Where $\underbrace{\partial_x X_0}_{\text{homogeneous}} = \underbrace{\partial_t X_0}_{\text{stationary}} = 0$, where X is any variable.

Let's also consider $\beta = P_{turb}/P_{mag} \ll 1$ appropriate for magnetised molecular clouds.

Continuity Eq.:

nonlinear .... bye!

$$\frac{\partial(\rho_0+\delta\rho)}{\partial t} + \nabla\cdot\left[(\rho_0+\delta\rho)\,\delta v\right] = 0$$

$$\boxed{\frac{\partial\,\delta\rho}{\partial t} + \nabla\cdot(\rho_0\,\delta v) = 0}$$

Momentum Eq.:

$$\frac{\partial(\rho_0+\delta\rho)\,\delta v}{\partial t} + \nabla\cdot\overbrace{(\rho_0+\delta\rho)\,\delta v\,\delta v}^{\text{non linear... bye!}}$$

$$= -\nabla c_s^2(\rho_0+\delta\rho) + \frac{1}{4\pi}\underbrace{\left[\nabla\times(B_0+\delta B)\right]\times(B_0+\delta B)}$$

$$\nabla\times(B_0+\delta B)]\times(B_0+\delta B)$$

$$= \left[\cancel{\nabla\times B_0}^{\,0} + \nabla\times\delta B\right]\times(B_0+\delta B)$$

$$= \underline{(\nabla\times\delta B)\times B_0} + \underbrace{(\nabla\times\delta B)\times\delta B}_{\substack{\text{nonlinear}\\ \sim\,\delta B^2/L}}$$

$$\boxed{\rho_0\frac{\partial\,\delta v}{\partial t} = -c_s^2\nabla\delta\rho + \frac{1}{4\pi}(\nabla\times\delta B)\times B_0}$$

Induction Eq.:

$$\frac{\partial(B_0+\delta B)}{\partial t} = \nabla\times\left[\delta v \times (B_0+\delta B)\right]$$

$$\boxed{\frac{\partial \delta B}{\partial t} = \nabla\times(\delta v\times B_0)}$$

Linearised MHD Eqs.:

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot(\rho_0\delta v) = 0$$ lin. continuity

$$\rho_0\frac{\partial \delta v}{\partial t} = -c_s^2\nabla\delta\rho + \frac{1}{4\pi}(\nabla\times\delta B)\times B_0$$ lin. momentum

$$\frac{\partial \delta B}{\partial t} = \nabla\times(\delta v\times B_0)$$ lin. induction

$$\nabla\cdot\delta B = 0$$ lin divergence.

Solvable with Fourier transform. Let's take the $\mathcal{F}\{f(x)\} := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$ of the linearized MHD equations.

Reminder:

1) $\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}f(x,t)\right\} = (ik)^n\tilde{f}(k,\omega)$

2) $\mathcal{F}\left\{ \frac{\partial^n}{\partial t^n} f(x,t) \right\} = (-i\omega)^n \tilde{f}(k,\omega)$

$a, b \in \mathbb{R}$

3) $\mathcal{F}\left\{ a f(x,t) + b g(x,t) \right\} = a\tilde{f}(k,\omega) + b\tilde{g}(k,\omega)$

Fourier transform is linear and turns derivatives into algebra.

Let's apply:

$$-\cancel{i}\omega\,\widetilde{\delta\rho} + \cancel{i}\rho_0 (k \cdot \widetilde{\delta v}) = 0$$

$$-\rho_0 \cancel{i}\omega\,\widetilde{\delta v} = -c_s^2 \cancel{i} k\,\widetilde{\delta\rho} + \frac{1}{4\pi}(\cancel{i}k \times \widetilde{\delta B}) \times B_0$$

$$-\cancel{i}\omega\,\delta B = \cancel{i}k \times (\widetilde{\delta v} \times B_0)$$

$$\cancel{i}k \cdot \delta B = 0$$

$$-\omega\,\widetilde{\delta\rho} + \rho_0 (k \cdot \widetilde{\delta v}) = 0$$ lin k-space continuity

$$-\rho_0 \omega\,\widetilde{\delta v} = -c_s^2 k\,\widetilde{\delta\rho} + \frac{1}{4\pi}(k \times \widetilde{\delta B}) \times B_0$$ lin k-space momentum

$$-\omega\,\delta B = k \times (\widetilde{\delta v} \times B_0)$$ lin k-space induction

$$k \cdot \delta B = 0$$ lin. k-space divergence-free.

Choose wave geometry :

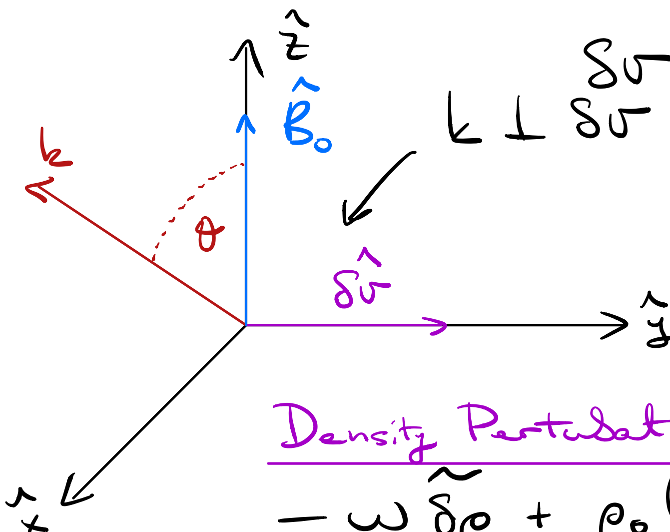
let $k = (k_x, 0, k_z)$, $B_0 = B_0 \hat{z}$

This is just for pedagogy but always have freedom to pick coordinates.

Transverse waves:

$$\delta v = (0, \delta v_y, 0)$$

$k \perp \delta v$



Density Perturbation:

$$-\omega \tilde{\delta\rho} + \rho_0 (k \cdot \tilde{\delta v}) = 0$$

$$\tilde{\delta\rho} = \rho_0 \frac{k \cdot \tilde{\delta v}}{\omega}$$

Since $k \perp \delta v \Rightarrow k \cdot \tilde{\delta v} = 0$

$$\Rightarrow \tilde{\delta\rho} = 0$$

i.e. Transverse waves do not perturb density. $\Rightarrow$ incompressible.

Velocity and Magnetic Perturbations

linearised momentum:

$\boxed{\text{only } y\text{-component survives}}$

$$-\rho_0 \omega \widetilde{\delta v_y} = -\cancel{c_s^2 k \widetilde{\delta\rho}} + \frac{1}{4\pi}\left(k \times \widetilde{\delta B}\right) \times B_0$$

$$\rho_0 \omega \widetilde{\delta v_y} = \frac{1}{4\pi}\left(k \times \widetilde{\delta B}\right) \times B_0$$

$$\rho_0 \omega \widetilde{\delta v_y} + \frac{k_z B_0}{4\pi} \widetilde{\delta B_y} = 0$$

$\boxed{\text{only y-component survives}}$

linearised induction:

$$-\omega \widetilde{\delta B_y} = k \times \left(\widetilde{\delta v} \times B_0\right)$$

$$k_z B_0 \widetilde{\delta v_y} + \omega \widetilde{\delta B_y} = 0$$

Combine linearised equations:

$$\begin{bmatrix} \rho_0 \omega & \frac{k_z B_0}{4\pi} \\ k_z B_0 & \omega \end{bmatrix} \begin{bmatrix} \widetilde{\delta v_y} \\ \widetilde{\delta B_y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

only interested in non-trivial solutions $\widetilde{\delta v} \neq 0$, $\widetilde{\delta B} \neq 0$.

$$\Rightarrow \quad \det\left(\begin{bmatrix} \rho_0 \omega & k_z B_0/4\pi \\ k_z B_0 & \omega \end{bmatrix}\right) = 0$$

$\boxed{\text{Reminder}}$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} a_{22} - a_{12} a_{21}$$

Hence: $$\rho_0 \omega^2 - \frac{k_z^2 B_0^2}{4\pi} = 0$$

$$\omega^2 = k_z^2 \left(\frac{B_0^2}{4\pi \rho_0} \right)$$

$$\boxed{\omega^2 = k_z^2 v_{A0}^2}$$

where $v_{A0} = B_0 / \sqrt{4\pi\rho_0}$ is the Alfvén speed.

dispersion relation for Alfvén waves

The full relation is:

$$k_z = k \cos\vartheta$$

ϑ is angle between $\hat{k}$ and $\hat{B}_0$

$$\Rightarrow \quad \boxed{\omega^2 = k^2 v_{A0}^2 \cos^2\vartheta}$$

for planar waves of the form $A(x,t) = A_0 e^{i(kx-\omega t)}$

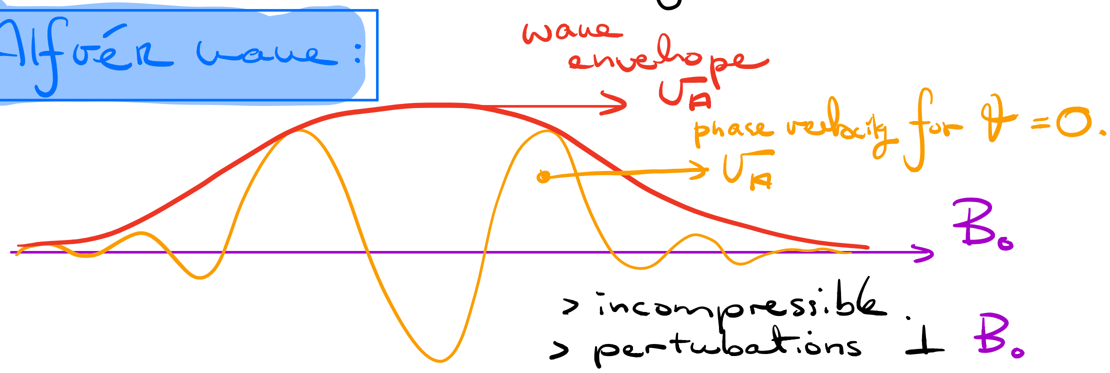
$\frac{d\omega}{dk}$:= group velocity $= v_{A0} \cos\vartheta$

$\frac{\omega}{k}$:= phase velocity $= v_{A0} \cos\vartheta$

independent of k

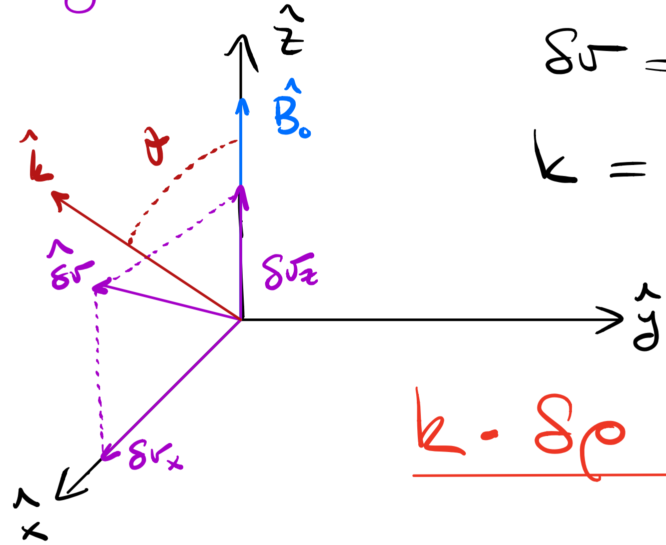
What we are describing is an

Alfvén wave:



> incompressible
> pertubations $\perp$ B_0

Longitudinal Waves



$$\delta v = (\delta v_x, 0, \delta v_z)$$

$$k = (k_x, 0, k_z)$$

$$k \cdot \delta \rho \neq 0 \, !$$

Perturbations

δB/

$$-\omega \tilde{\delta B} = k \times (\tilde{\delta v} \times B_0) \quad \text{lin k-space induction}$$

$$= k \times (-\tilde{\delta v}_x B_0)\, \hat{y}$$

$$\tilde{\delta B} = \frac{\tilde{\delta v}_x B_0}{\omega} (k \times \hat{y})$$

δv/, δB/

... leave for the reader

Dispersion relation:

same as previously but for $\tilde{\delta \rho}$, $\tilde{\delta B}$, $\tilde{\delta v}$. This is because $k \cdot \delta v \neq 0$.

This results in a 3x3 matrix problem of the form $\det(A) = 0$, with characteristic equation:

$$\omega^4 - \omega^2 k^2 (v_{A0}^2 + c_s^2) + k^2 k_z^2 v_A^2 c_s^2 = 0$$

which is a quadratic in $\omega^2 = \zeta$

$$\zeta^2 - k^2 (v_{A0}^2 + c_s^2)\zeta + k^4 \cos^2\theta \, v_{A0}^2 c_s^2 = 0$$

$$\omega_{1,2}^2 = \zeta_{1,2} = \frac{k^2}{2} v_{fast}^2 \pm \sqrt{\frac{1}{4} k^4 v_{fast}^4 - k^4 v_{A0}^2 c_s^2 \cos^2\theta}$$

where $v_{fast}^2 = v_{A0}^2 + c_s^2$.

Simplify and seperate roots

$$\omega_{1,2}^2 = k^2 \left(\frac{1}{2} v_{fast}^2 \pm \sqrt{\frac{v_{fast}^4}{4} - v_{A0}^2 c_s^2 \cos^2\theta} \right)$$

$$= \frac{k^2}{2} v_{fast}^2 \left(1 \pm 2 \sqrt{\frac{1}{4} - \frac{v_{A0}^2 c_s^2}{v_{fast}^4} \cos^2\theta} \right)$$

$$= \frac{k^2}{2} v_{fast}^2 \left[1 \pm \sqrt{1 - 4 \frac{v_{A0}^2 c_s^2}{v_{fast}^4} \cos^2\theta} \right]$$

Use binomial approximation

$$(1 + x)^{\alpha} \approx 1 + \alpha x$$

for $\left|\frac{v_{A0}^2 c_s^2}{(v_{A0}^2 + c_s^2)^2}\right| \ll 1$, $x = -4\frac{v_{A0}^2 c_s^2}{v_{fast}^4}\cos^2\theta$

$$\omega_{1,2}^2 \simeq \frac{k^2}{2} v_{fast}^2 \left[1 \pm \left(1 - 2\frac{v_{A0}^2 c_s^2}{v_{fast}^4}\cos^2\theta\right)\right]$$

Positive root

$$\omega^2 \simeq k^2 v_{fast}^2 - \frac{v_{A0}^2 c_s^2}{v_{fast}^4}\cos^2\theta$$

since $\left|\frac{v_{A0}^2 c_s^2}{v_{fast}^4}\right| \ll 1$ for $\beta \ll 1$ plasma

$$\boxed{\omega^2 \simeq k^2 v_{fast}^2}$$

dispersion relation for fast magnetosonic wave.

> compressible, $k \cdot \delta v \neq 0$.
> maximised at $v_{fast} \perp B_0$.

group velocity = phase velocity = v_{fast}

negative root

$$\omega^2 \simeq \frac{k^2}{2} v_{fast}^2 \left[1 - \left(1 - 2\frac{v_{A0}^2 c_s^2}{v_{fast}^4}\cos^2\theta\right)\right]$$

$$\simeq k^2 \frac{v_{A0}^2 c_s^2}{v_{A0}^2 + c_s^2} \cos^2\theta$$

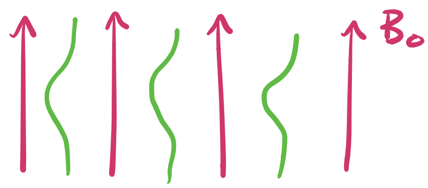
$$\omega^2 \simeq k^2 \left(c_s^2 - \frac{c_s^4}{v_{A0}^2 + c_s^2} \right) \cos^2\theta$$

$\left| c_s^4 / (v_{A0}^2 + c_s^2) \right| \ll 1$ because $v_{A0} \gg c_s$ for $\beta \ll 1$.

$$\boxed{\omega^2 \simeq k^2 c_s^2 \cos^2\theta}$$

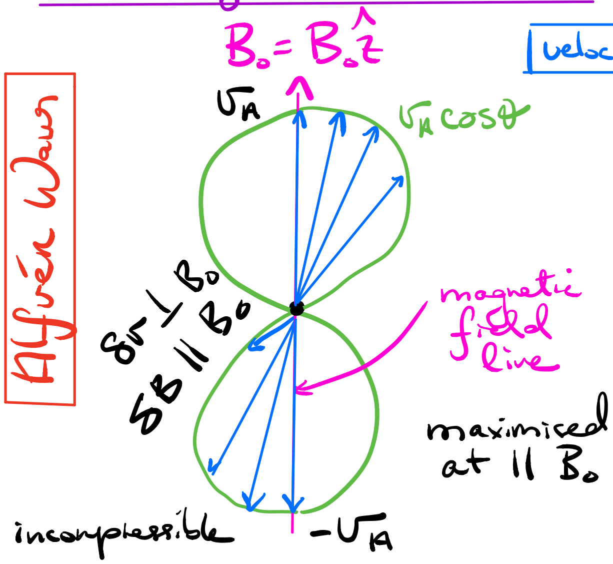
dispersion relation for slow wave.

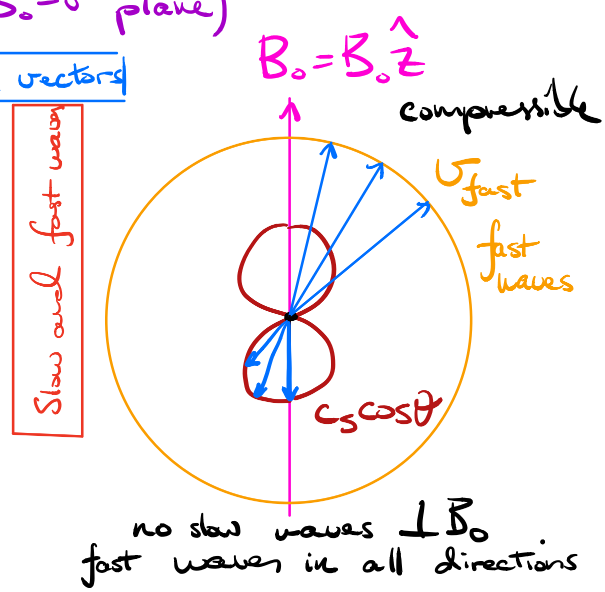
> compressible.
> travels at $c_s \parallel B_0$.
> do not travel $\perp B_0$.



group velocity = phase velocity $= c_s \cos\theta$

Summary of 3-waves ($B_0 - v$ plane)





Note:

1) linearised MHD equations not good for supersonic turbulence.
e.g. $|\delta\rho^2| \ll 1$ to be ignored. However $|\delta\rho| \sim \mathcal{M}^2 \Rightarrow |\delta\rho^2| \sim \mathcal{M}^4 \Rightarrow$ non-linear terms matter.

2) MHD turbulence has been historically framed as a wave-interaction phenomena between Alfvén wave packets.
Fundamentally this ignores other waves, however we a learning of their importance in the last few years.