ASTR4012/ASTR8002
Astrophysical Gas Dynamics
Assignment 4 – exam assignment

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due Tuesday, October 13, 2020
(no extensions possible – exam assignment)

1 Validity of the gas/fluid approximation

In the following, calculate and discuss the conditions for approximating a collection of particles as a gas or fluid. Order-of-magnitude calculations are sufficient.

(a) Calculate the particle mean free path (λ) of air on Earth (typical density of air near the surface, typical temperature $\sim 300\,\mathrm{K}$, etc.). Write down the size scales over which air can be treated as a gas and over which it cannot.

(2 points)

(b) What is particle mean free path in the solar corona (density $\rho \sim 10^{-16}\,\mathrm{g\,cm^{-3}}$)? Start from this density and make use of the mean particle weight in the solar corona. Briefly discuss (2–3 sentences) what the main difference is in computing λ for air and λ in the solar corona.

(2 points)

(c) Calculate the mean free path in a galaxy cluster, i.e., in the intra-cluster medium (ICM). Compare this mean free path to the typical size of a galaxy cluster and discuss briefly (again 2–3 sentences) the validity of the fluid/gas approximation in the ICM.

(2 points)

(d) What is the particle mean free path in a molecular cloud? Compare two cases: (1) assuming collisions of molecular hydrogen molecules if there are no charges and/or no magnetic fields present, and (2) for the case where the cross section for collisions would be primarily determined by electrostatic interactions.

(1 point)

(e) What would you do if you want to describe the dynamics of a system in which particle collisions are extremely rare events? Only write 1–2 sentences.

(1 point)

2 Rankine-Hugoniot shock jump conditions

(a) Find the compression ratio $r \equiv \rho_2/\rho_1$ at a plane shock front in terms of the upstream Mach number $\mathcal{M} = v_1/c_1$, where $c_1 = (\gamma P_1/\rho_1)^{1/2}$ is the upstream sound speed (subscripts 1 and 2 refer to the pre-shock and post-shock quantities, respectively). Assume that the gas flows along the shock normal.

Hint: Use the hydrodynamic equations in conservation form to identify the flux of mass, momentum and energy entering and leaving the shock front.

(5 points)

(b) A supernova remnant shock moves at $\sim$ 5,000 km/s into a part of the interstellar medium with temperature of $\sim 10^4$ K. Estimate the downstream (post-shock) temperature, assuming the medium consists of ionised hydrogen in full thermodynamic equilibrium.

(3 points)

(c) What would be the approximate temperature behind a shock front moving at 50,000 km/s, if full thermodynamic equilibrium were established? Why does this approximation fail?

(2 points)

3 Evolution of a supernova remnant

Assume first the supernova remnant in its adiabatic phase: all the mass of the remnant is concentrated in a thin shell located at the position of the shock at radius $r = r_s(t)$, where $M \approx (4\pi/3)r^3\rho_1$, with ρ_1 being the density of the ambient (interstellar) medium. Furthermore, the pressure $P(t)$ interior to the shock can be considered uniform and the equation of motion for the thin shell is given by

$$\frac{d}{dt}(M\dot{r}_s) = 4\pi r_s^2\, P, \tag{1}$$

where $\dot{r}_s = dr_s/dt$.

(a) Use the jump conditions for a strong shock of an ideal gas with an adiabatic index $\gamma = 5/3$ to estimate the thickness Δr of the shell in terms of r_s (assume the shell density equals the post-shock density).

(2 points)

(b) Given that in the Sedov phase, the total internal energy of the gas in the remnant equals 80% of the explosion energy E_{SN}, show that the equations of motion have a solution of the form

$$r_s = At^\alpha. \tag{2}$$

Find the constants A and α.

(4 points)

(c) Assume now that the shell cools rapidly. Because the cooling rate of the gas is proportional to the density squared, there is a phase in the evolution when the thermal energy of the freshly shocked gas can no longer be shared evenly throughout the remnant, as it was the case in the energy-driven phase discussed above. Most of the energy is radiated away and a cool dense shell forms around the still hot interior. To a good approximation, the interior can be described as a hot adiabatic gas bubble of constant mass with an equation of state as before, $P \propto \rho^\gamma$ with $\gamma = 5/3$. The evolution of the blast wave is now driven by the adiabatic expansion of the bubble. Show that this pressure-driven 'snow plow phase' admits again a solution of the form

$$r_s \propto t^\beta, \tag{3}$$

and find the index β.

Hint: the equation of motion of the shell, Eq. (1), still applies in this phase.

(4 points)

4 Magnetic breaking of a rotating gas disc

Consider a uniform gaseous disc of density $\rho_{\rm cl}$ and half-thickness Z rotating rigidly with an initial angular velocity Ω_0. Furthermore assume the disc is threaded by a magnetic field $\mathbf{B}$ of strength B_0, initially uniform and parallel to the rotation axis of the disc. The magnetic field couples the disc ($|z| \leq Z$) with the external medium ($|z| > Z$) of density $\rho_{\rm ext}$, which is initially at rest. Use cylindrical coordinates (R, ϕ, z), assume symmetry around the rotation axis, i.e., $\partial\phi = 0$, and assume ideal MHD.

(a) For this system, derive the evolution equation for the angular velocity Ω, where $v_\phi = R\Omega$, and the evolution equation for the toroidal component of the magnetic field, B_ϕ. In your derivations, assume that $v_R \ll v_\phi$, $v_z \ll v_\phi$, and $B_R \ll B_0$, allowing you to drop 2nd-order terms in the Euler and induction equation. The final results should be

$$\frac{\partial \Omega}{\partial t} = \frac{B_0}{4\pi R\,\rho}\,\frac{\partial B_\phi}{\partial z}, \quad \text{and} \tag{4}$$

$$\frac{\partial B_\phi}{\partial t} = B_0 R\,\frac{\partial \Omega}{\partial z}. \tag{5}$$

(4 points)

(b) Show that the evolution of the external medium can be expressed by the wave equation

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\rm A,ext}^2\,\frac{\partial^2 \Omega}{\partial z^2}, \tag{6}$$

with the Alfvén speed $v_{\rm A,ext} = B_0/(4\pi\rho_{\rm ext})^{1/2}$ in the external medium.

(2 points)

(c) Derive the evolution equation of the angular velocity at the surface of the disc ($|z| = Z$), using the torque per unit area $N = RB_0B_\phi/(4\pi)$ that the magnetic field exerts on the surface of the disc. The result should be

$$\frac{\partial^2 \Omega_{\rm cl}}{\partial t^2} = \frac{1}{Z}\,\frac{\rho_{\rm ext}}{\rho_{\rm cl}}\,v_{\rm A,ext}^2\,\left.\frac{\partial \Omega}{\partial z}\right|_{|z|=Z} \tag{7}$$

Hint: The full torque is $I\partial\Omega/\partial t$ with the moment of inertia I for a cylinder.

(3 points)

(d) Combine Equations (6) and (7) to calculate the spin-down time of the disc.

Hint: Use the solution of Equation (6) at the disc surface and make the ansatz $\Omega \propto \exp(-\alpha t + \beta z)$ to determine the constants α and β.

(3 points)

Please submit your solutions via Turnitin by the assignment deadline. **Please note that this is the exam assignment, so absolutely no extensions are possible!**