ASTR4012/ASTR8002
Astrophysical Gas Dynamics
Assignment 3

Christoph Federrath

due Friday, September 25, 2020

1 Parker wind solution

Consider a steady, radial, polytropic flow of gas in the gravitational field of a star with mass M_*. The polytropic equation of state is

$$P = K\rho^{\Gamma}, \tag{1}$$

where K is constant along the streamlines and $\Gamma < 5/3$.

1. Show that the continuity equation can be written as

   $$\dot{M} = 4\pi r^2 \rho v, \tag{2}$$

   where $v(r)$ is the radial velocity and $\dot{M}$ is the constant change of mass.

   (5 points)

2. Derive the relevant Euler equation for this spherically symmetric system. Eventually, this should only be expressed as a function of radius r, velocity v, sound speed c_{s}, and gravitational potential $-GM_*/r$ of the star.

   (10 points)

3. Show that a smooth solution containing both sub- and supersonic regions of the flow exists only if

   $$r_{\mathrm{s}} = \frac{GM_*}{2c_{\mathrm{s}}^2} \quad \text{at} \quad v^2 = c_{\mathrm{s}}^2, \tag{3}$$

   where $c_{\mathrm{s}} = (\Gamma P/\rho)^{1/2}$ is the sound speed (which depends on r).

   (5 points)

4. Imposing this condition, and the boundary conditions

   $$\rho = \rho_* \quad \text{and} \quad c_{\mathrm{s}} = c_* \tag{4}$$

   at the surface $r = r_*$ of the star, find the mass loss rate $\dot{M}$ in the wind, in the limit of low surface velocity $v_* \ll c_*$, given that the surface temperature is large compared to the virial temperature, i.e., $c_*^2 \gg GM_*/r_*$.

   Hint: Find the mass loss rate at the sonic point r_{s}. Start by integrating the Euler equation over radius and determine the integration constant by inserting the respective values at the sonic point. Express the final answer in terms of stellar parameters and gas parameters only; this should lead to

   $$\dot{M} = \pi M_*^2 G^2 (\Gamma K)^{-3/2} \left[\rho_* \left(\frac{2}{5-3\Gamma}\right)^{1/(\Gamma-1)}\right]^{(5-3\Gamma)/2}. \tag{5}$$

   (10 points)

5. Show that under the same approximations as in the previous question, the location of the sonic point expressed in terms of sound speed at the stellar surface is

$$r_{\mathrm{s}} = \frac{GM_*}{2c_*^2}\frac{5-3\Gamma}{2}, \tag{6}$$

and discuss the behaviour of the solutions as Γ goes to 5/3.

(5 points)

2 Lane-Emden equation

In a spherical symmetric system the equations of hydrostatic equilibrium and Poisson's equation are

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{d\Phi}{dr}, \tag{7}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho, \tag{8}$$

where Φ is the gravitational potential and G is Newton's constant.

1. Taking $\Phi(r_*) = 0$ and $\rho(r_*) = 0$ at the surface of the star ($r = r_*$), show that for a polytropic equation of state, $P = K\rho^\Gamma = K\rho^{(n+1)/n}$, the density in the star ($\Phi < 0$) can be expressed as

$$\rho = \left(\frac{-\Phi}{K(n+1)}\right)^n. \tag{9}$$

(5 points)

2. Substitute this expression into the Poisson equation and show that this then reduces to the *Lane-Emden equation*,

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = 0, \tag{10}$$

when written in terms of the dimensionless variables $w = \Phi/\Phi_c$ and $z = r/r_0$, where Φ_c is the potential at $r = 0$ and r_0 is a characteristic radius. Find an expression for r_0 in terms of n and K.

(5 points)

3. Given that there exists a solution of the Lane-Emden equation for the given system, show that the radius r_* of a non-relativistic degenerate star ($n = 3/2$) is related to its total mass M_* by

$$r_* \propto M_*^{-1/3}. \tag{11}$$

(5 points)

Please submit your solutions via Turnitin by the assignment deadline.