ASTR4012/ASTR8002
Astrophysical Gas Dynamics
Assignment 2

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1 Pressure of an ideal gas

The Maxwell-Boltzmann distribution of a thermal gas is given by

$$f_{\mathrm{MB}}(v)\,d^3v = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{mv^2}{2k_{\mathrm{B}}T}\right)d^3v, \tag{1}$$

where m is the mass of each particle, n the number density, v the velocity, T the temperature, and k_{B} is the Boltzmann constant. Show that the pressure is given by

$$P = nk_{\mathrm{B}}T, \tag{2}$$

by using the following integral that defines the pressure from the distribution function,

$$P = \frac{4\pi}{3m}\int_0^\infty p^4 f(p)dp, \tag{3}$$

where $p = mv$ is the momentum of each particle. *Hint: get started by transforming the velocity distribution to the momentum distribution, by using the principle of probability conservation.*

(5 points)

2 Parker Instability

Consider an isothermal gas in the galactic disc, which is threaded by a horizontal magnetic field. Assume a constant gravitational field perpendicular to the disc plane in the z direction, i.e., $\mathbf{g} = -g\hat{z}$, and a magnetic field parallel to the disc plane along x, which only varies with z, i.e., $\mathbf{B} = B(z)\hat{x}$. For simplicity, study the system in two dimensions (x and z) using cartesian coordinates.

(a) Assume that the system is in hydrostatic equilibrium with a constant ratio of the magnetic to thermal pressure, i.e.,

$$\alpha \equiv \frac{B^2}{8\pi P} = \mathrm{const.} \tag{4}$$

What is the pressure distribution as a function of z? Use the isothermal gas relation $P = c_{\mathrm{s}}^2\rho$, where c_{s} is the constant sound speed and the scale height $H = (1+\alpha)c_{\mathrm{s}}^2/g$ to express the result.

Hint: The Euler equation with a magnetic field reads

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi\rho}(\mathbf{B}\cdot\nabla)\mathbf{B}. \tag{5}$$

(Have a thought about the additional terms compared to without B fields.)

(5 points)

(b) Now consider this system slightly perturbed out of its equilibrium. Then, from the linear perturbation analysis, one gets the following dispersion relation in the xz-plane,

$$n^4 + c_s^2\left[(1+2\alpha)\left(k^2 + \frac{k_0^2}{4}\right)\right]n^2 + k_x^2 c_s^4\left[2\alpha k^2 + k_0^2\left[\left(1+\frac{3\alpha}{2}\right) - (1+\alpha)^2\right]\right] = 0, \tag{6}$$

where $n \equiv i\omega$ and $k_0 \equiv H^{-1}$, and the Fourier modes in the x and z direction for the perturbed variables are proportional to

$$e^{i(\omega t - k_x x)}, \quad \text{and} \quad e^{i(\omega t - k_z z)}, \tag{7}$$

with $k^2 = k_x^2 + k_z^2$. (Feel free to derive the dispersion relation, Eq. 6, yourself from the magnetohydrodynamical equations in the relevant geometry.)

Show that in the absence of a magnetic field all roots (in terms of n^2) of this dispersion relation are negative, i.e., $n^2 < 0$. What is the physical implication of this result regarding instability?

(5 points)

(c) In the case of a non-vanishing magnetic field, derive the instability criterion for the Parker instability (magnetic Rayleigh-Taylor instability),

$$\left(\frac{2k}{k_0}\right)^2 < 2\alpha + 1. \tag{8}$$

Hint: use the roots of n^2 to find at least one unstable mode, i.e., $n^2 > 0$.

(5 points)

(d) Show that the instability criterion, Equation (8), is equivalent to

$$\lambda_x > \Lambda_x \equiv 4\pi H\left(\frac{1}{2\alpha+1}\right)^{1/2} \tag{9}$$

and

$$\lambda_z > \Lambda_z \equiv \frac{\Lambda_x}{[1 - (\Lambda_x/\lambda_x)^2]^{1/2}}, \tag{10}$$

with the wavelengths $\lambda_x = 2\pi/k_x$ and $\lambda_z = 2\pi/k_z$.

(5 points)

Please submit your solutions via Turnitin by the assignment deadline.